Lecture 1: From data to paradox

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Formal Theories of Vagueness
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Course Team

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Course Aim

This course has two main aims.

- **Introduce vagueness:** the Sorites paradox, tolerance, and vagueness as central phenomena in natural language and in the world.
- **Introduce philosophical logic:** use vagueness as a case study for comparing formal systems beyond classical logic.

Materials

- The slides are the primary study material.
- `https://theories-of-vagueness.github.io/`
- *Suggested* readings will be provided for each class.

Tentative schedule

- **Monday:** Vagueness and the Sorites paradox
- **Tuesday:** Many-valued logics
- **Wednesday:** Supervaluationism and subvaluationism
- **Thursday:** Epistemicism and contextualism, and intro to Strict/tolerant logics
- **Friday:** Strict/tolerant logics and recent approaches

Today's Readings

- MacFarlane, John. 2021. "Vagueness and the Sorites Paradox." Chapter 8 of *Philosophical Logic: A Contemporary Introduction*, 191–214. Routledge.
- van Rooij, Robert. 2011. "Vagueness and linguistics." In *Vagueness: A Guide*, 123–170. Springer.
- Kooiman, Isa, and Robert van Rooij. 2025. "Dissecting two problems of vagueness." *Analysis* 85(4): 848–858.

Outline

Red

Please scan the QR code

https://tinyurl.com/2026red

Is this red? (1)

Is this red? (1)

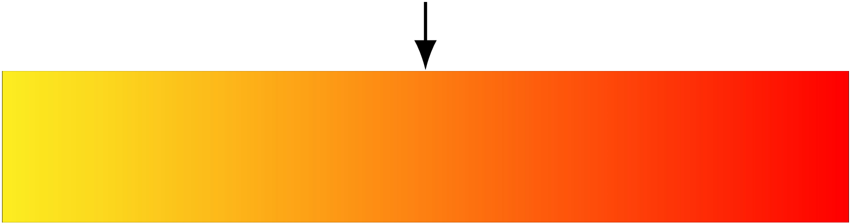
Is this red? (2)

Is this red? (2)

Is this red? (3)

Is this red? (3)

Is this red? (4)



Is this red? (4)

Is this red? (5)

Is this red? (5)

Is this red? (6)

Is this red? (6)

This is red

This is red

So this is red

So this is red

So this is red

So this is red

So this is red

So this is red

So this is red

So this is red

...

...

So this is red!

So this is red!

Red, tall, and bald

A term is **vague** when its correct application admits *indeterminate/borderline cases* and obeys a *tolerance* idea (small changes shouldn't flip the verdict). So there is no sharp cutoff.

- *Red*: along a hue continuum there is a range where it's unclear whether a patch is red.
- *Tall*: there is a grey zone where neither "tall" nor "not tall" seems clearly correct.
- *Bald*: there is no exact hair count at which someone becomes bald. Adding or removing one hair shouldn't change the judgment.

Distinguishing vagueness: Ambiguity

Is vagueness ambiguity?

There is a duck by the BANK.

bank = financial institution *or* riverbank.

- Ambiguity = one expression with *multiple conventional meanings*.
- Disambiguation (by context or paraphrase) selects **a single meaning**.
- **No borderline** cases are required for ambiguity.

Distinguishing vagueness: Context-dependence

Does vagueness = context-dependence?

Not necessarily. Some terms are context-dependent but *crisp*, others remain *vague* even after context is fixed.

- Context-dependent but crisp (not vague): *I*, *today*, *here*.
- Vague & context-dependent: *tall* (threshold varies by group/standard, yet a grey zone remains).
- Vague but not allegedly context-dependent: *bald*, *heap*, *bush*.

Distinguishing vagueness: Underspecification

Is vagueness just underspecification?

Underspecification = missing detail where a precise value still exists.

- Underspecified (not vague): "The meeting is *sometime this afternoon*." (An exact time exists but isn't given.)
- Vague: *red*, *heap*, *bald*: small changes shouldn't flip the verdict (tolerance), so there are indeterminate cases.

Underspecification is about *information left open*. Vagueness is about *tolerance* in meaning.

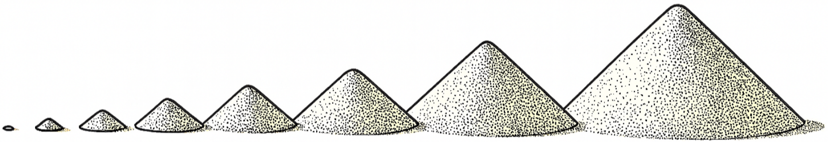
Criteria for Vagueness

- Lack of sharp boundaries
- Presence of indeterminate cases
- Tolerant to small differences along a relevant dimension
- Can lead to the Sorites paradox

The Sorites paradox

1 million grains make a heap.
If 1M grains make a heap, then (1M-1) grains make a heap.
If (1M-1) grains make a heap, then (1M-2) grains make a heap.
. . .
If 2 grains make a heap, then 1 grain makes a heap.

Therefore, 1 grain makes a heap.



Abstract sorites schema

Many-premise (downward from a large N, e.g. 1M):

$$
\begin{array}{l}
P(N) \\
P(N) \to P(N-1) \\
P(N-1) \to P(N-2) \\
\vdots \\
P(2) \to P(1) \\
\hline
\therefore\ P(1)
\end{array}
$$

Inductive (quantified, downward):

$$
\begin{array}{l}
P(N) \\
\forall n\,(P(n) \to P(n-1)) \\
\hline
\therefore\ \forall k\,(1 \leq k \leq N \Rightarrow P(k)) \text{ (in particular } P(1)\text{).}
\end{array}
$$

Similarity

We can also formulate sorites arguments using a similarity relation.

- Let $a_1, \ldots, a_n$ be a sorites series for predicate P.
- Write $a_i \sim a_{i+1}$ when adjacent cases are relevantly similar.
- $P(a_1)$: the first case clearly satisfies P.
- $\neg P(a_n)$: the last case clearly does not satisfy P.
- The tension is between local similarity and different endpoint verdicts.

Tolerance-premise version

$$\begin{array}{l}
P(a_1) \\
a_1 \sim a_2 \\
a_2 \sim a_3 \\
\vdots \\
a_{n-1} \sim a_n \\
\forall x \forall y ((P(x) \wedge x \sim y) \rightarrow P(y)) \\
\hline
\therefore\ P(a_n)
\end{array}$$

Premises

- The premises $P(n) \to P(n-1)$ can also be expressed as

$$\neg(P(n) \wedge \neg P(n-1))$$

- "It is not the case that n grains make a heap while $n-1$ grains do not."
- Or as

$$\neg P(n) \vee P(n-1)$$

- "Either n grains do not make a heap, or $n-1$ grains make a heap."
- The negated conjunction expresses a constraint: there is no sharp cutoff
- The conditional and the disjunction seem to require a positive verdict about the neighbouring case.

The Sorites paradox

$$
\begin{array}{l}
P(N) \\
P(N) \rightarrow P(N-1) \\
P(N-1) \rightarrow P(N-2) \\
\vdots \\
P(2) \rightarrow P(1) \\
\hline
\therefore\ P(1)
\end{array}
$$

- We have many plausible premises (local tolerance steps)
- and one rule: **Modus Ponens.**
- What to give up?

Tuesday: Many-valued logics / LP

- Classical sorites reasoning assumes each $P(n)$ is either true or false.
- Many-valued logics add a value for indeterminate cases.
- Gappy option: some $P(n)$ is neither true nor false.
- Glutty option: some $P(n)$ is both true and false.

Wednesday: Supervaluationism / Subvaluationism

- The sorites treats P as if it already had a precise extension.
- Supervaluationism considers all admissible precisifications of P.
- A is supertrue iff true on all admissible precisifications.
- Borderline cases are neither supertrue nor superfalse.
- Subvaluationism uses the dual idea: A is subtrue iff true on at least one admissible precisification.
- The paradox is blocked by replacing a single extension with a space of admissible sharpenings.

Thursday: Epistemicism / Contextualism

- Epistemicism accepts a sharp cutoff: for some n, $P(n)$ is true and $P(n-1)$ is false.
- The cutoff is unknowable, so tolerance feels compelling even though it is false.
- Contextualism instead says the standard for P can shift during the argument.
- Each local judgement may seem acceptable in its own context.
- But the full sorites chain does not preserve one fixed standard.
- The paradox is blocked either by hidden ignorance or by contextual shift.

Friday: The logic of ST

- ST distinguishes strict truth from tolerant truth.
- Borderline cases can be tolerantly true without being strictly true.
- This makes room for local tolerance: nearby cases may be treated alike.
- ST can preserve Modus Ponens for local reasoning.
- But ST consequence is not unrestrictedly transitive.
- The paradox is blocked because acceptable local steps cannot simply be chained from $P(N)$ to $P(1)$.

Outline

1. Sorites Paradox
2. Vagueness in Language
3. Vagueness in the World

Vagueness in language

- Adjectives: *tall*, *expensive*, *red*, *flat*.
- Nouns: *heap*, *child*, *mountain*, *chair*.
- Verbs: *run*, *start*, *finish*, *understand*.
- Quantifiers: *many*, *few*, *a lot*.
- Prepositions: *near*, *around*, *behind*.

Vagueness and gradability

The main linguistic case study is **gradable adjectives**.

- Gradable adjectives order objects along a dimension.
- *Tall* orders objects by height.
- *Expensive* orders objects by cost.
- *Red* orders colour patches by similarity to focal red.
- Comparatives make the ordering explicit: *taller than*, *more expensive than*, *redder than*.

Relative adjectives

- Relative adjectives have **context-dependent standards**.
- Examples: *tall*, *short*, *expensive*, *cheap*, *long*.
- Their positive form depends on what counts as high or low enough in the context.
- *This coffee is expensive* may be true relative to ordinary coffee prices.
- The same sentence may be false relative to airport prices.
- Relative adjectives naturally give rise to borderline cases.
- They also naturally generate sorites arguments.

Absolute adjectives

- Absolute adjectives are associated with distinguished **endpoints on a scale**.
- Examples: *full*, *empty*, *dry*, *wet*, *straight*, *bent*.
- Their positive form is often tied to an endpoint on a scale.
- *full* tends toward a maximum standard.
- Their standards are therefore more closely constrained by scale structure than those of relative adjectives.
- For instance:
  - *The rod is long* depends heavily on a contextual standard.
  - *The rod is straight* is much closer to an endpoint condition.
- But endpoint orientation does not eliminate vagueness. A glass may count as *full* despite containing a tiny amount of empty space.

Multidimensionality

Multidimensionality interacts with vagueness, but should be distinguished from it.

- One adjective may be evaluated along different dimensions.
- *London is larger than New York* may mean larger by population or larger by area.
- Once the relevant dimension is fixed, vagueness can remain.

Using vague language

A use of a vague expression can affect the conversational context.

- Suppose everyone knows Feynman's height.
- If someone says *Feynman is tall*, the utterance may not add information about Feynman.
- Instead, it may tell us something about the current standard for *tall*.
- The utterance can sharpen the context: standards that are too high are no longer live options.

This gives vague predicates a metalinguistic role:

- they describe the world
- and they also help negotiate how words are being used.

[Barker 2002]

Why is language vague?

Vagueness may be useful, not merely defective.

- It allows flexible application across different comparison classes.
- It reduces cognitive and communicative costs.
- It lets speakers communicate when exact measurements are unavailable.
- It leaves room for negotiation, discretion, and future adjustment.
- It can express evaluative information, not just measurement.

Outline

Ontological vagueness

and the Problem of the Many

Isa Kooijman & Robert van Rooij, ‘Dissecting two problems of vagueness, Analysis, 2025

The Sorites paradox

- A man of 200 cm heigh is tall
- If a man of 200 cm is tall, then a man of 199 cm is tall.
- A man of 199 cm heigh is tall
- ...
- A man of 141 cm heigh is tall
- If a man of 141 cm is tall, then a man of 140 cm is tall.
- A man of 140 cm heigh is tall.
- $\Rightarrow$ Every man is tall

- Can go other way around as well
- $\Rightarrow$ Every man is tall and not tall.

A realistic picture

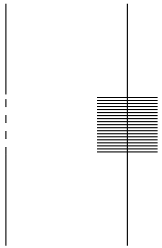


Figure: A Sorites sequence with its vague boundary region (left) and a rough representation of the potential precise cut-off points (right).

The Problem of the Many

- Consider Kilimanjaro, K.
- For many rocks near the border of K, it is not clear whether they are part of K.
- $\Rightarrow K$ does not seem to have a sharp boundary.
- There are many equally good ways to draw a precise boundary of K.
- To each way corresponds a precise aggregate of rocks.
- These are all equally good candidates to be K.
  It is arbitrary to pick one as K.
- If all count as K, then we have not one but many K's
- If none count as K, then we have not one but no K's.
- Either way, there is not one K.

Unger on the Problem of the Many

[...] for our new problem, the leading idea is to focus on physical, spatial situations where no natural boundaries, no natural stopping places, are to be encountered. (Unger, 1980, The Problem of the Many)

Lewis's conclusion

Once noticed, we can see that [the Problem of the Many] is everywhere, for all things are swarms of particles. There are always outlying particles, questionably part of the thing, not definitely included and not definitely not included. So there are always many aggregates, differing by a little bit here and a little bit there, with equal claim to be the thing. We have many things or we have none, but anyway not the one thing we thought we had. That is absurd.
(Lewis, 1993, Many, but almost one)

Vagueness of Classification and Individuation

The generally accepted understanding of the two puzzles

- Vagueness of Classification: vagueness of how a certain object or aggregate of particles should be classified into (linguistic) concepts.
- Classification $\Rightarrow$ the Sorites paradox
- Vagueness of Individuation: vagueness of how to individuate objects
- The apparent vague spatial boundary of objects
- Individuation $\Rightarrow$ the Problem of the Many
- Suggestion (by quotes Unger and Lewis)
  1. Individuation of objects $\Leftrightarrow$ the Problem of the Many
  2. Classification into concepts $\Leftrightarrow$ the Sorites paradox
- Problems of different nature: ontology vs. semantics
- But the relation is more complex....

The Sorites paradox for individuation of objects

- Consider Kilimanjaro, K.
- r$_m$ is part of K.
- If r$_m$ is part of K, then the rock next to it, r$_{m+1}$, is part of K.
- r$_{m+1}$, is part of K
- If r$_{m+1}$ is part of K, then the rock next to it, r$_{m+2}$, is part of K.
- ...
- If r$_{m+(n-1)}$ is part of K, then the rock next to it, r$_{m+n}$ is part of K.

  Therefore,
- r$_{m+n}$ is part of K.

- $\Rightarrow$ all rocks are part of K
- and not part of K.

The Sorites paradox for objects

- So, we have the Sorites paradox for the individuation of objects

- Do we also have the Problem of the Many for Classification?

The Problem of the Many for classification into concepts, 1

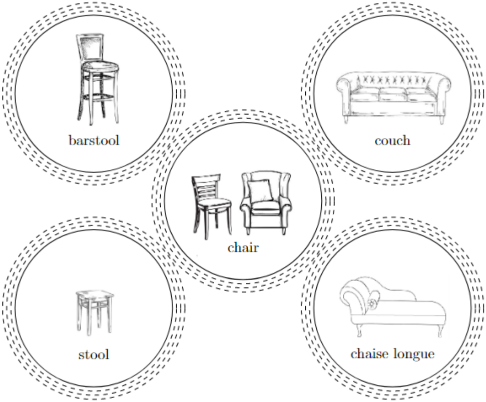


Figure: The chair spectrum.

The Problem of the Many for classification into concepts, 1

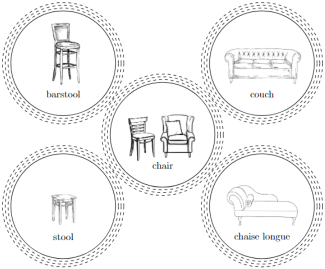


Figure: The chair spectrum.

1. measure of legs. → barstool	where to draw the line?
2. width of seating → couch	where to draw the line?
3. measure of back → stool	where to draw the line?
4. length of seeting → chaise longue	where to draw the line?

The Problem of the Many for classification into concepts, 2

Figure: The colour spectrum.

Three dimensions

1. Hue
2. Saturation
3. Lightness

The new puzzle

- The Sorites paradox occurs both for
  1. Classification into concepts, and
  2. Identification of objects

- The Problem of the Many occurs both for
  1. Identification of objects, and
  2. Classification into concepts

- But what, then, is the relation between the two puzzles?

Identification leads to Many Sorites sequences

- Any physical object has many spatial boundaries
- For each boundary, it is unclear where the exact boundary point is
- Each boundary leads to a Sorites paradox
- So, any physical object leads to many Sorities paradoxes
- $\boxed{\text{Problem of the Many} \Rightarrow \text{Many Sorites paradoxes}}$

In a picture

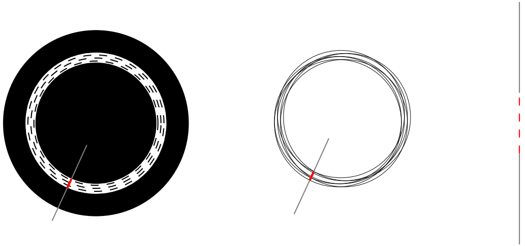


Figure: A round object with its vague boundary (left) and a rough representation of the potential precise boundaries (middle) giving rise to the Problem of the Many. Each of the boundary regions (coloured red) potentially gives rise to its own Sorites sequence and corresponding Sorites paradox (right).

A second picture, for concepts

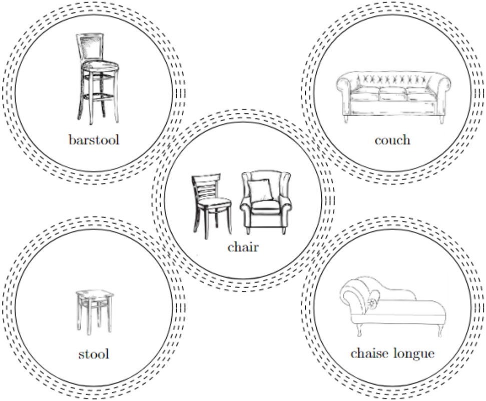


Figure: The chair spectrum.

Classification: $\Diamond$ many, always $\exists$ Sorites sequence(s)

- Concepts can give rise to many Sorites sequences
- Typical for multi-dimensional concepts
- Chair, Red, Intelligent, etc.

- but some concepts are one-dimensional
- Tall, loud, hard, ...
- These only give rise to one Sorites sequence.

Remaining issues

- How to deal with Sorites sequences of individuation?
- Perhaps vague objects?
- But then no Problem of the Many (but Evans' problem)
- How to deal with Problem of the Many for categorization?
- Or is this a problem of identification (of properties 'Chair' and 'Red') after all?
- But then it becomes an ontological problem

Conclusion

- Relation Sorites and Problem of the Many is not
  1. Classification into concepts $\Leftrightarrow$ the Sorites paradox
  2. Individuation of objects $\Leftrightarrow$ the Problem of the Many
- Rather:
  $\boxed{\text{Problem of the Many = Many Sorites sequences/paradoxes}}$
- However: if vague objects exists $\Rightarrow$ no Problem of the Many
- So, above identification only if $\boxed{\text{constitution is determinate}}$
- What this means for classification and individuation is open

Nature of Vagueness

- Russell: All vagueness is linguistic

  Still dominant position

- But then: All necessity is verbal necessity

  was also once dominant position

- Now popular: different types of necessity
- Also different types of vagueness?
- or do they have a common core?

Vagueness in the mind, for cognitive reasons

- Traditional: Vagueness is linguistic or ontic
- Claim: Vagueness is primarily in the mind
- Similarity/soft boundaries already crucial for cognitive reasons
  1. Agents want to **learn** generalisations (crucial for life) based on relations between stimuli and effect

     Much better if assumed that similar stimuli have similar effects
  2. ML $\Rightarrow$ don't **represent** concepts by hard thresholds
     hard threshold functions not differentiable (gradient descent)

     Instead use logistic function of z: $\frac{1}{1+e^z}$

     $\Rightarrow$ Gives rise to smooth thresholds = vague boundaries
  3. Via communication by bounded rational agents

     "Vagueness, Signalling and Bounded Rationality", (Franke & Jäger & van Rooij, 2010)
- $\Rightarrow$ Vagueness in mind for cognitive reasons

Conclusions

- We have to give up:
  1. Sorities Problem $\Leftrightarrow$ Classification (semantics)
  2. Problem of the Many $\Leftrightarrow$ Identification (ontology)
- Or more semantics, or more ontology
- But actually, vagueness is best explained as being in the mind
- For cognitive reasons