Lecture 2: Many-valued logics

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Formal Theories of Vagueness
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Plan

1. From Sorites to Many Values
2. Three-valued Logics
3. Sorites
4. n-valued logics
5. Logic of Paradox
6. Higher-order Vagueness

Today's Readings

Suggested:

- Priest, Graham. *An Introduction to Non-Classical Logic*, ch. 7.1–7.5.
- Hájek, Petr. *Metamathematics of Fuzzy Logic*

Outline

1. From Sorites to Many Values
2. Three-valued Logics
3. Sorites
4. n-valued logics
5. Logic of Paradox
6. Higher-order Vagueness

Recall: the Sorites pressure

$$
\begin{array}{l}
P(N) \\
P(N) \to P(N-1) \\
P(N-1) \to P(N-2) \\
\vdots \\
P(2) \to P(1) \\
\hline
\therefore\ P(1)
\end{array}
$$

- The endpoint premise is compelling: a million grains make a heap.
- Each local tolerance step seems compelling: one grain should not matter.
- The conclusion is unacceptable: one grain makes a heap.
- Classical reasoning turns many small steps into a problematic long chain.

The many-valued idea

- Classical logic assumes that every relevant sentence is simply true or false.
- Indeterminate cases are not naturally described as simply true or simply false.
- Add a **third value** i for indeterminate cases.
- Or add **many intermediate values** between 0 and 1.
- The hope: represent gradual transition.

The extra value

Reading	Interpretation of i
Gap	neither true nor false: indeterminate, undefined, unsettled
Glut	both true and false: overdetermined, inconsistent, dialetheic
Degree	half-true: a point on a scale between full falsity and full truth

Outline

Classical semantics

Let P be the set of propositional letters. A classical valuation is a map $v : P \to \{0, 1\}$ extended recursively:

$$\begin{aligned} v(\neg\varphi) &= 1 - v(\varphi) \\ v(\varphi \wedge \psi) &= \min(v(\varphi), v(\psi)) \\ v(\varphi \vee \psi) &= \max(v(\varphi), v(\psi)) \\ v(\varphi \to \psi) &= \max(1 - v(\varphi), v(\psi)) \end{aligned}$$

$\Gamma \models \varphi$ iff every valuation that makes all premises true also makes the conclusion true.

Logical matrices

A finite many-valued logic is specified by a **logical matrix**:

1. a finite non-empty set of truth values T
2. a set $T^+ \subseteq T$ of *designated* values
3. one truth-function for each connective

$$\Gamma \models \varphi$$

iff every valuation that makes all premises in Γ designated also makes φ designated. That is:

$$\forall v, \text{if } v(\gamma) \in T^+ \text{ for every } \gamma \in \Gamma, \text{ then } v(\varphi) \in T^+$$

For most of this section: $T = \{0, i, 1\}$ and $T^+ = \{1\}$

What can vary?

- The value set can vary.
- The designated set can vary: only 1, or also intermediate values.
- The truth-functions can vary: different negations, conjunctions, disjunctions, implications.
- The notion of logical consequence.

Truth-tables: open choices

$\wedge$	1	i	0
1	1	$?$	0
i	$?$	$?$	$?$
0	0	$?$	0

$\vee$	1	i	0
1	1	$?$	1
i	$?$	$?$	$?$
0	1	$?$	0

$\to$	1	i	0
1	1	$?$	0
i	$?$	$?$	$?$
0	1	$?$	1

$\neg$	
1	0
i	$?$
0	1

Take $\wedge$. How many truth value functions can you define? $3^5 = 243$

Some natural constraints:

Idempotence: $p \wedge p \equiv p$

Symmetry: $p \wedge q \equiv q \wedge p$

How many truth value functions for $\wedge$? $3^2 = 9$

Strong Kleene K_3^s

$\wedge$	1	i	0
1	1	i	0
i	i	i	0
0	0	0	0

$\vee$	1	i	0
1	1	1	1
i	1	i	i
0	1	i	0

$\rightarrow$	1	i	0
1	1	i	0
i	1	i	i
0	1	1	1

$\neg$	
1	0
i	i
0	1

- i behaves like indeterminacy or partial information.
- Return i only when the classical truth-conditions do not already settle the value.
- $i \wedge 0 = 0$ because one false conjunct suffices for falsity.
- $i \vee 1 = 1$ because one true disjunct suffices for truth.
- $\neg i = i$ (negation doesn't resolve indeterminacy).
- $i \rightarrow i = i$, since $\rightarrow$ is treated as $\neg p \vee q$.

Practice: computing $p \leftrightarrow q$ in K_3^s

We define the biconditional by combining two conditionals:

$$p \leftrightarrow q := (p \to q) \land (q \to p)$$

$p \to q$	$q = 1$	$q = i$	$q = 0$
$p = 1$	1	i	0
$p = i$	1	i	i
$p = 0$	1	1	1

$p \leftarrow q$	$q = 1$	$q = i$	$q = 0$
$p = 1$	1	1	1
$p = i$	i	i	1
$p = 0$	0	i	1

$p \leftrightarrow q$	$q = 1$	$q = i$	$q = 0$
$p = 1$	1	i	0
$p = i$	i	i	i
$p = 0$	0	i	1

Strong Kleene by numbers

Read i as $\frac{1}{2}$ and order the values:

$$0 < \frac{1}{2} < 1$$

Then Strong Kleene has the same numerical clauses as classical logic:

$$p \wedge q = \min(p, q) \qquad p \vee q = \max(p, q)$$
$$\neg p = 1 - p \qquad p \rightarrow q = \max(1 - p, q)$$

Thus, K_3^s conservatively extends the classical truth-functions. More precisely, for every K_3^s-valuation v and formula φ:

$$v(\varphi) = \frac{1}{2} \quad \text{if the classical completions of } v \text{ disagree on } \varphi$$

Classicality preservation

Lemma

Let $\mathrm{Var}(\varphi)$ be the set of propositional variables occurring in φ. If $v(p) \in \{0, 1\}$ for every $p \in \mathrm{Var}(\varphi)$, then in K_3^s:

$$v(\varphi) \in \{0, 1\}$$

and the value agrees with the classical value of φ.

Proof idea. Structural induction on φ.

- Atoms are classical by assumption.
- Each Strong Kleene connective agrees with its classical counterpart on inputs in $\{0, 1\}$.
- Hence no occurrence of i can be generated from purely classical atoms.

Weak Kleene K_3^w

$\wedge$	1	i	0
1	1	i	0
i	i	i	i
0	0	i	0

$\vee$	1	i	0
1	1	i	1
i	i	i	i
0	1	i	0

$\rightarrow$	1	i	0
1	1	i	0
i	i	i	i
0	1	i	1

$\neg$	
1	0
i	i
0	1

- i behaves like meaninglessness or undefinedness.
- Infectiousness: if a complex formula contains an undefined component, the whole formula is undefined.
- Examples: failed presuppositions, undefinedness, NaN.

Some immediate facts

- K_3^s and K_3^w have no tautologies.
- Reason: if every atom receives i, every formula receives i.
- But their consequence relations are not trivial.

A divergence:

$$p \models_{K_3^s} p \vee q \qquad \text{but} \qquad p \not\models_{K_3^w} p \vee q$$

Why? Take $v(p) = 1$, $v(q) = i$. In K_3^w, $v(p \vee q) = i$, not designated.

Jan Łukasiewicz (1878-1956)

- Polish logician and philosopher, a leading figure of the Lwów-Warsaw school.
- Pioneered many-valued logic and introduced Polish (prefix) notation.
- He was motivated by Aristotle's problem of *future contingents* (e.g., "There will be a sea battle tomorrow").
- Such future-tensed statements seem neither true nor false *now*.
- Łukasiewicz proposed a third value for the present status of such claims.

Jan Łukasiewicz, 1878-1956

Łukasiewicz three-valued logic Ł3

$\wedge$	1	i	0
1	1	i	0
i	i	i	0
0	0	0	0

$\vee$	1	i	0
1	1	1	1
i	1	i	i
0	1	i	0

$\rightarrow$	1	i	0
1	1	i	0
i	1	$\mathbf{1}$	i
0	1	1	1

$\neg$	
1	0
i	i
0	1

- Conjunction and disjunction behave like in K_3^s.
- The key difference is implication: $i \rightarrow i = 1$.
- Motivation: in classical logic $p \rightarrow p$ is always valid. Łukasiewicz wanted to preserve this tautology even with a third value.
- But note: $p \rightarrow q \not\equiv \neg p \vee q$.

Modality in Ł3

$\to$	1	i	0
1	1	i	0
i	1	1	i
0	1	1	1

$\neg$	
1	0
i	i
0	1

Find formulas built from $\neg$ and $\to$ with the following values:

p	$\Diamond p$	Δp	$\mathsf{I}p$
1	1	1	0
i	1	0	1
0	0	0	0

$\Diamond p := \neg p \to p \qquad \Delta p := \neg(p \to \neg p) \qquad \mathsf{I}p := (\neg p \to p) \wedge (p \to \neg p)$

What survives classically?

Let $\mathcal{T}_{CL}$ and $\mathcal{T}_{\text{Ł3}}$ be the sets of tautologies of classical logic and Ł3.

$$\mathcal{T}_{\text{Ł3}} \subsetneq \mathcal{T}_{CL}$$

- Every Ł3 tautology is classically valid, because Ł3 agrees with classical logic on classical valuations.
- The inclusion is proper. An example? $p \vee \neg p$

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The Sorites revisited

Recall the structure of the Sorites paradox:

p_N
$p_N \to p_{N-1}$
$\vdots$
$p_2 \to p_1$

$\therefore\ p_1$

p_N
$p_N \to p_{N-1}$
$\dots$

$p_{k-1} \to p_k$
$p_k \to p_{k-1}$
$\dots$
$p_l \to p_{l-1}$

$\dots$
$p_2 \to p_1$

$\therefore\ p_1$

- Using classical logic (left), the conclusion must be true.
- Using K_3^s (right), we can make some of the premises neither true nor false, and the conclusion false.
- What would be the situation with Ł3?

The K_3^s answer

The K_3^s answer to the sorites paradox is: **we reject some of the premises as not true.**

Some problems

- **Tolerance vs. local failure:** Three-valued logics block Sorites by letting some tolerance links be *indeterminate*. Yet ordinary intuition treats each link as (seems) true. Revise logic, or revise the intuition?
- **Arbitrariness:** The boundary between 1 and i and i and 0 feels arbitrary and not precise.

The problem of penumbral connections

- Let C be the sentence "Alex is a child" and let A be the sentence "Alex is an adult".
- Suppose Alex is an adolescent and an indeterminate case of both predicates, so $v(C) = v(A) = i$.
- Nevertheless, being a child and being an adult are mutually exclusive. Intuitively, we therefore want $v(C \wedge A) = 0$.
- Conjunction should also be idempotent. Since $v(C) = i$, we want $v(C \wedge C) = v(C) = i$.
- But a truth-functional conjunction considers only the values of its inputs. Consequently, $v(C \wedge A) = f_\wedge(i, i)$ and $v(C \wedge C) = f_\wedge(i, i)$.
- Truth values alone do not represent the semantic connection between *child* and *adult*: although both predicates are indeterminate, they remain incompatible.
- Fine (1975) calls such relations between vague predicates **penumbral connections**.

Outline

n-valued logics: why go beyond three?

- We may replace the single value i (or $\frac{1}{2}$) with multiple intermediate degrees to achieve finer granularity.
- Sorites links evaluate *close to* 1 rather than landing at $\frac{1}{2}$.
- At each step, the truth degree drops slightly, matching the "small change, small effect" intuition.
- Our task is to extend a 3-valued matrix to an n-valued semantics.

Recalling definitions

- Given a language $\mathcal{L}$, the general make-up of a finite many-valued logic will be formed by
  1. A finite non-empty set of truth values T
  2. A set $T^+ \subseteq T$ of designated truth values
  3. For each n-place connective, a truth value function $v : T^n \to T$. If $n = 0$, $v(\cdot) \in T$
- These three elements form the **logical matrix** of $\mathcal{L}$.
- Given a set of formulas Γ and a formula ϕ, we say that Γ **entails** ϕ (written $\Gamma \models \phi$) iff for every valuation v, whenever $v(\gamma) \in T^+$ for all $\gamma \in \Gamma$, it follows that $v(\phi) \in T^+$

n-valued logics

- For $n \geq 2$, an n-valued logic is defined over the set of truth-values:

$$T_n = \left\{ \tfrac{k}{n-1} : k = 0, 1, \ldots, n-1 \right\} \subseteq [0, 1]$$

- We take $T^+ = \{1\}$.
- Different truth-value functions for the connectives give rise to different logics.
- In particular, we can extend K_3^s and Ł3 to K_n^s and Łn by extending pointwise the same semantic clauses (e.g., conjunction defined as the minimum of the values).

Three implications

Read values as $0 < \frac{1}{2} < 1$.

$$\begin{aligned}
(K_3^s): &\quad x \to y = \max(1-x, y) \\
(\text{Ł}3): &\quad x \Rightarrow y = \min(1, 1-x+y) \\
(\star): &\quad x \to_\star y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{if } x > y. \end{cases}
\end{aligned}$$

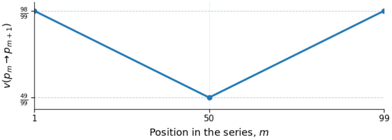
- Strong Kleene: material implication under partial information.
- Łukasiewicz: full truth whenever the consequent is at least as true as the antecedent.
- $\star$: no compensation. If the consequent falls below the antecedent, the conditional takes the consequent's value.

Sorites in n-valued Strong Kleene

Let $v(p_m) = 1 - \frac{m-1}{99}$ for $m = 1, \ldots, 100$ as positions in the series.
So values decline from 1 to 0 in equal steps.

For $m = 1, \ldots, 99$:

$$\begin{aligned} v(p_m \to p_{m+1}) &= \max(1 - v(p_m), v(p_{m+1})) \\ &= \max\left(\frac{m-1}{99}, 1 - \frac{m}{99}\right) \end{aligned}$$



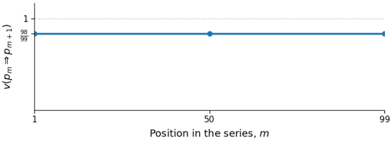
- The first link has value $\frac{98}{99}$.
- Values dip toward the middle of the series.
- The final links rise again.
- No link has value 1, so no tolerance premise is fully true.

Sorites with Łukasiewicz implication

Use the same profile: $v(p_m) = 1 - \frac{m-1}{99}$.

For $m = 1, \ldots, 99$:

$$\begin{aligned} v(p_m \Rightarrow p_{m+1}) &= \min\left(1, 1 - v(p_m) + v(p_{m+1})\right) \\ &= \min\left(1, 1 - \left[1 - \frac{m-1}{99}\right] + \left[1 - \frac{m}{99}\right]\right) \\ &= 1 - \frac{1}{99} = \frac{98}{99} \end{aligned}$$



- Every local step is nearly true.
- But every local step is still less than fully true.
- The argument is still blocked because the premises are not all designated.

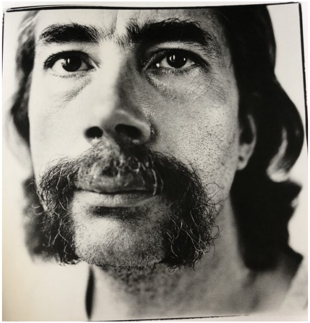
Discussion

- How many values n? Equal spacing or fitted curves? What fixes the step? The sharp line is replaced by parameters that still need justification.
- Do we really want *degrees of truth*? Or are we modeling degrees of belief/evidence/assertability instead?
- Are degrees commensurable across predicates? e.g. is *Amsterdam is beautiful* "truer" than *New York is big*? If comparability fails, a single numerical scale for all sentences may mislead.

[Williamson 2002]

Outline

The logic of paradox (LP)



Graham Priest

- We now consider a different answer to the Sorites: the premises are true, but *Modus Ponens* is not a valid rule of inference.
- We have always assumed that $T^+ = \{1\}$.
- The Logic of Paradox (LP) takes $T^+ = \{1, i\}$ with Strong Kleene semantics.

[Priest 2008]

Paraconsistent vs Paracomplete

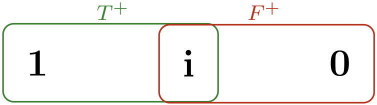
- A logic with consequence relation $\models$ is *paraconsistent* iff there exist formulas ϕ, ψ such that $\{\phi, \neg\phi\} \not\models \psi$.
- A logic with $\models$ is *paracomplete* iff there exists a formula ϕ such that $\not\models \phi \vee \neg\phi$.
- LP (with $T^+ = \{1, i\}$ and Strong Kleene tables) is paraconsistent: $\{p, \neg p\} \not\models_{\text{LP}} q$.
- K_3^s (with $T^+ = \{1\}$) is paracomplete: $\not\models_{K_3^s} p \vee \neg p$

LP: gaps and gluts

- Let T^+/F^+ be the set of truth/falsity designated values.
- ***Gap:*** a value t is a gap iff $t \notin T^+$ and $t \notin F^+$
- ***Glut:*** a value t is a glut iff $t \in T^+$ and $t \in F^+$

- In K3, i is a *gap* (neither true nor false).

- In LP, i is a *glut* (both true and false).



LP and the Sorites

Modus Ponens fails: $p,\ p \to q\ \not\models_{\mathsf{LP}}\ q$

The Sorites argument relies on repeated applications of Modus Ponens:

$$P(N), \quad P(N) \to P(N-1), \quad P(N-1) \to P(N-2), \quad \ldots$$

In LP, the local tolerance premises may all be designated:

$$P(k) \to P(k-1) \in T^{+}_{\mathsf{LP}}$$

but they cannot be chained by unrestricted Modus Ponens:

$$P(k),\ P(k) \to P(k-1) \not\models_{\mathsf{LP}} P(k-1)$$

LP blocks the Sorites by rejecting the validity of the inferential chain, not by rejecting the local tolerance premises.

Classical Tautologies

The set of tautologies in LP coincides with the set of classical tautologies.

Theorem (LP-Classical Validity Equivalence)

Let LP be the propositional logic using the Strong Kleene truth-functions on $\{0, i, 1\}$ with designated values $T^+_{\mathsf{LP}} = \{1, i\}$. Then for every propositional formula ϕ,

$$\models_{\mathsf{LP}} \phi \text{ iff } \models_{\mathsf{CL}} \phi$$

Lemma 1: Refinement

Lemma 1 (Refinement)

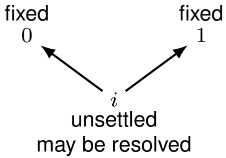
Let $v, w : P \to \{0, i, 1\}$, using the Strong Kleene truth-functions. Write $v \leq w$ iff w keeps all settled atomic values fixed:

$$v(p) = 0 \Rightarrow w(p) = 0 \qquad v(p) = 1 \Rightarrow w(p) = 1$$

Then for every formula ϕ:

$$v(\phi) \in \{0, 1\} \quad \Longrightarrow \quad w(\phi) = v(\phi)$$

$v(p)$	0	i	1
$w(p)$	0	0 or i or 1	1



- Kleene tables do not give a settled verdict that can later be overturned by resolving i-values.
- If $v(\phi) \in \{0, 1\}$, then the value of ϕ is already forced by the settled info in v.
- But not in Ł3: $i \to_{Ł3} i = 1$
- A classical refinement can make the same conditional false!

Lemma 2: Classicality

Lemma 2 (Classicality)

Let $v : P \to \{0, 1\}$ be a *classical* valuation. Write v_{SK} for the extension using the Strong Kleene truth-functions and v_{CL} for the classical (two-valued) extension. Then, for every formula ϕ,

$$v_{\mathrm{SK}}(\phi) = v_{\mathrm{CL}}(\phi)$$

Proof idea: countermodels move both ways

$$\models_{\mathsf{LP}} \phi \quad \text{iff} \quad \models_{\mathrm{CL}} \phi$$

From LP to CL

$$\not\models_{\mathrm{CL}} \phi$$

$$\Downarrow$$

$$\exists v\ v_{\mathrm{CL}}(\phi) = 0$$

$\Downarrow$ Lemma 2

$$v_{\mathsf{LP}}(\phi) = 0$$

$$\Downarrow$$

$$\not\models_{\mathsf{LP}} \phi$$

From CL to LP

$$\not\models_{\mathsf{LP}} \phi$$

$$\Downarrow$$

$$\exists v\ v_{\mathsf{LP}}(\phi) = 0$$

$\Downarrow$ refine the valuation

$$\begin{array}{ccc} 0 & i & 1 \\ \downarrow & \swarrow\ \searrow & \downarrow \\ 0 & 0 \text{ or } 1 & 1 \end{array}$$

$\Downarrow$ Lemma 1: the value 0 is preserved

$$\exists w \geq v\ w_{\mathrm{SK}}(\phi) = 0$$

$\Downarrow$ Lemma 2: classical valuations agree

$$w_{\mathrm{CL}}(\phi) = 0 \quad \Rightarrow \quad \not\models_{\mathrm{CL}} \phi$$

Assessing the Situation

The Ł$_n/K_n^s$ answer to the sorites paradox is: **we reject some of the premises as not true.**

The LP answer to the sorites paradox is: **the argument is not valid (modus ponens fails).**

Outline

Higher-order vagueness: the core worry

- Three-valued accounts soften the sharp $1/0$ cut by adding a indeterminate value i.
- But then two interfaces remain: $1/i$ and $i/0$.
- When we talk about what is *definitely* true/false, do these interfaces themselves become sharp again?

$$\mathbf{1} \quad \underset{\text{interface}}{\Big|} \quad \mathbf{i} \quad \underset{\text{interface}}{\Big|} \quad \mathbf{0}$$

A crisp determinacy operator

Interpret "definitely" as an operator Δ in K_3^s:

$$v(\Delta\varphi) = \begin{cases} 1 & \text{iff } v(\varphi) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We define first-order indeterminate status as:

$$\nabla\varphi := \neg\Delta\varphi \wedge \neg\Delta\neg\varphi$$

p	Δp	∇p
1	1	0
i	0	1
0	0	0

Collapse of higher-order vagueness

$$\Delta\Delta\varphi \leftrightarrow \Delta\varphi$$

$$\nabla^{(1)}\varphi := \neg\Delta\varphi \wedge \neg\Delta\neg\varphi$$
$$\nabla^{(m)}\varphi := \nabla(\Delta^{m-1}\varphi) \qquad (m \geq 2)$$

- First-order vagueness can remain: $\nabla^{(1)}\varphi$ may be true.
- But second-order vagueness collapses: $\nabla^{(2)}\varphi$ is never true.
- The account gives sharp boundaries for determinacy itself.

What many-valued approaches buy us

- A formal status for indeterminate cases.
- A way to block the classical Sorites argument.
- A distinction between full truth, partial truth, and designated truth.
- Fine-grained modeling of gradual change.
- A bridge to paraconsistent and paracomplete logics.

Main costs

- Why exactly these truth values?
- Why exactly these truth-functions?
- Do degrees model truth, assertability, evidence, or belief?
- Are degrees comparable across predicates?
- Does the approach solve higher-order vagueness, or merely relocate it?

Wednesday: supervaluationism and subvaluationism

- Instead of adding extra truth values, keep classical truth at each precise sharpening.
- A vague predicate has many admissible precisifications.
- Supertruth: true on all admissible precisifications.
- Subtruth: true on at least one admissible precisification.
- Can we handle vagueness with many classical sharpenings rather than many truth values?

Outline

Truth and Degrees[1]

- We introduced n-valued logics using a finite set of truth values. Truth-value domains need *not* be finite:
  - Finite-valued:
  $$T_n = \{0, \tfrac{1}{n-1}, \tfrac{2}{n-1}, \ldots, 1\},\ n \in \mathbb{N}, n \geq 2$$
  - Rational-valued:
  $$T_{\aleph_0} = \{\tfrac{m}{n} : 0 \leq m \leq n,\ m, n \in \mathbb{N},\ n \neq 0\}$$
  - Real-valued:
  $$T_{\aleph_1} = [0, 1]$$
- To get a many-valued logic, we choose:
  - a truth-value set (finite or continuous);
  - how connectives act on degrees (e.g. $\wedge = \min$, $\vee = \max$, $\neg x = 1 - x$, and a suitable $\rightarrow$)
  - truth-preserving vs. degree-preserving logical consequence.

[1] We are making an important assumption in the notation used in this slide. Which one? ;-)

Fuzzy Logics and Logical Consequence

Given a set of truth values T with $1 \in T$:

Truth-preserving consequence $\models_1$

$$\Gamma \models_1 \psi \quad \text{iff} \quad \forall v \, (\forall \gamma \in \Gamma, \; v(\gamma) = 1) \Rightarrow v(\psi) = 1.$$

Intuition: preserves absolute truth (1).

Degree-preserving consequence $\models_{\text{deg}}$

$$\Gamma \models_{\text{deg}} \psi \iff \forall t \in T \; (\forall v[(\forall \gamma \in \Gamma, \; v(\gamma) \geq t) \;\Rightarrow\; v(\psi) \geq t])$$

Intuition: reasoning must hold *uniformly at every level of certainty*.

A simple case

Let $T = \{0, \frac{1}{2}, 1\}$. Use Strong Kleene semantics.

Truth-preserving ($\models_1$):

$$\Gamma \models_1 \psi \quad \Longleftrightarrow \quad \Gamma \models_{K_3^s} \psi$$

Degree-preserving ($\models_{\mathrm{deg}}$):

$$\Gamma \models_{\mathrm{deg}} \psi \quad \Longleftrightarrow \quad \Gamma \models_{\mathsf{LP}} \psi \text{ and } \Gamma \models_{K_3^s} \psi$$

Thresholds $t \in \{0, \frac{1}{2}, 1\}$; $t = 0$ vacuous, $t = \frac{1}{2}$ matches LP's $T^+ = \{\frac{1}{2}, 1\}$, and $t = 1$ matches K_3^s's $T^+ = \{1\}$

Modus Ponens

Modus Ponens under $\models_{\mathrm{deg}}$ over $T = [0, 1]$ fails

- *Strong Kleene $\rightarrow$:* take $t = 0.5$, $v(A) = 0.5$, $v(B) = 0.25$. Then $v(A \rightarrow B) = \max(1 - 0.5,\ 0.25) = 0.5 \geq t$ but $v(B) = 0.25 < t$.
- *Łukasiewicz $\Rightarrow$:* take $t = 0.5$, $v(A) = 0.5$, $v(B) = 0.25$. Then $v(A \Rightarrow B) = \min(1,\ 1 - 0.5 + 0.25) = 0.75 \geq t$ but $v(B) = 0.25 < t$.

Degree-preserving entailment as an inf-bound over $T = [0, 1]$

Fact (assume $\Gamma \neq \varnothing$)

$$\Gamma \models_{\mathrm{deg}} \varphi \quad \Longleftrightarrow \quad \forall v\ (\inf\{\, v(\psi) \mid \psi \in \Gamma \,\} \ \leq\ v(\varphi))$$

Recall (degree-preserving): $\Gamma \models_{\mathrm{deg}} \varphi$ iff for all valuations v and all thresholds $t \in [0, 1]$:

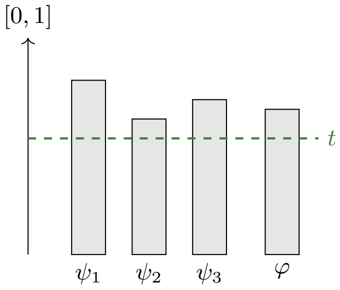
$$(\forall \gamma \in \Gamma \ \ v(\gamma) \geq t) \implies v(\varphi) \geq t$$

Remark. If you adopt $\inf \varnothing = 1$, the equivalence also holds for $\Gamma = \varnothing$ and yields $\varnothing \models_{\mathrm{deg}} \varphi$ iff $v(\varphi) = 1$ for all v.

Graphical intuition: thresholds vs. the infimum bar

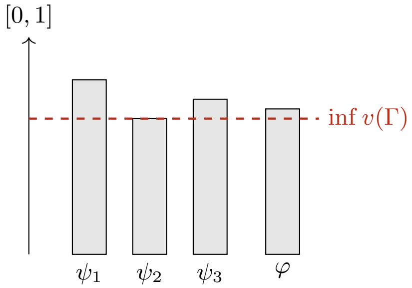
Threshold view:

- Draw any horizontal threshold t.
- If all premise bars are at/above t, then the conclusion bar must also be at/above t.



Infimum view:

- Let the red dashed line be $s = \inf v(\Gamma)$.
- Then $v(\varphi)$ must be at least s.



Fuzzy Logics and the Sorites

- Fix a finite chain $N \geq 2$ and choose any monotone profile $f : \{1, \ldots, N\} \to [0, 1]$ with $f(1) = 1 > \cdots > f(N) = 0$. Set $v(p_m) := f(m)$.
- **Truth-preserving** ($\models_1$): under Strong Kleene and Łukasiewicz semantics, each tolerance link is < 1.

The $\models_1$ answer to the sorites paradox is: **we reject some of the premises as not true**.

- **Degree-preserving** ($\models_{\mathrm{deg}}$): under Strong Kleene and Łukasiewicz semantics, MP fails.

The $\models_{\mathrm{deg}}$ answer to the sorites paradox is: **the argument is not valid (modus ponens fails)**.

The issue of higher-order vagueness

Fuzzy logic gives many degrees between 0 and 1.

But higher-order vagueness concerns the status of expressions like:

- definitely true
- definitely definitely true
- indeterminate definitely true

So we need a fuzzy semantics for a determinacy operator: $\Delta\phi$

Crisp determinacy

One possible operator is the crisp determinacy:

$$v(\Delta\varphi) = \begin{cases} 1 & \text{iff } v(\varphi) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then as in finite case: $\Delta\Delta\varphi \leftrightarrow \Delta\varphi$

- $\Delta\varphi$ is always either 0 or 1
- So there are no indeterminate cases of being definitely true.
- Higher-order vagueness collapses.

A continuous alternative: very true

Hájek treats fuzzy truth-values such as “very true” as unary connectives.

$$v(\Delta\varphi) = v(\varphi)^2$$

It satisfies $x^2 \leq x$, $0^2 = 0$, and $1^2 = 1$

So $\Delta\varphi \to \varphi$ is valid.

- If φ is very true, then φ is true.
- If $0 < v(\varphi) < 1$, then $\Delta\varphi$ is still intermediate, but lower.

[Hájek 2001]

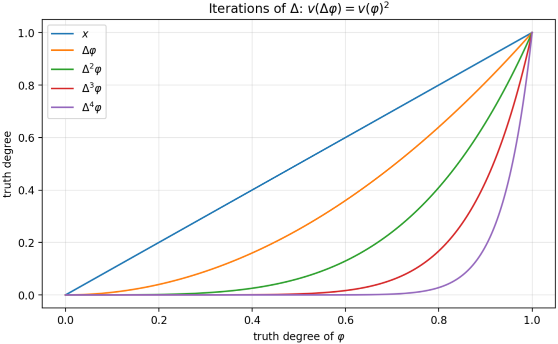
Iteration of very true

If $v(\Delta\varphi) = v(\varphi)^2$, then $v(\Delta^n\varphi) = v(\varphi)^{2^n}$

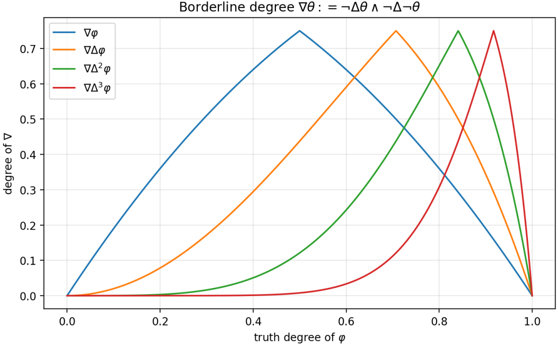
For $0 < x < 1$:

$$x > x^2 > x^4 > x^8 > \cdots > 0$$

- There is no immediate collapse: $\Delta\Delta\varphi \not\equiv \Delta\varphi$
- Every finite iteration remains intermediate.
- But the values converge to 0.



Iterations of Δ: $v(\Delta\varphi) = v(\varphi)^2$
x
$\Delta\varphi$
$\Delta^2\varphi$
$\Delta^3\varphi$
$\Delta^4\varphi$
truth degree
truth degree of φ
0.0
0.2
0.4
0.6
0.8
1.0



Borderline degree $\nabla\theta := \neg\Delta\theta \wedge \neg\Delta\neg\theta$
$\nabla\varphi$
$\nabla\Delta\varphi$
$\nabla\Delta^2\varphi$
$\nabla\Delta^3\varphi$
degree of ∇
truth degree of φ
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.0
0.2
0.4
0.6
0.8
1.0

Principles

It validates principles of the following form:

$$\Delta\varphi \to \varphi$$

$$\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$$

$$\Delta(\varphi \vee \psi) \leftrightarrow (\Delta\varphi \vee \Delta\psi)$$

[Hájek 2001]

Some failures

1. **B-like principle fails**: $\not\models \varphi \to \Delta\neg\Delta\neg\varphi$
   Let $v(\varphi) = 0.1$. Then:
   $$v(\Delta\neg\Delta\neg\varphi) = (2x - x^2)^2 = 0.0361$$
   so:
   $$v(\varphi \to \Delta\neg\Delta\neg\varphi) = 0.9361 < 1$$
2. **Deduction theorem fails**: $p \models_1 \Delta p \quad$ but $\quad \not\models_1 p \to \Delta p$
   For $v(p) = \frac{1}{2}$:
   $$v(p \to \Delta p) = \min(1, 1 - \tfrac{1}{2} + \tfrac{1}{4}) = \tfrac{3}{4} < 1$$
3. **Contraposition fails** $\psi \models_1 \neg p \quad$ but $\quad p \not\models_1 \neg\psi$
   (Exercise: find the relevant ψ)

Problems for this account

- Very true vs. definitely true: $v(\Delta\varphi) = v(\varphi)^2$ is naturally read as "φ is very true", not as "φ is definitely true".
- No fully true indeterminate cases: if $\nabla\varphi := \neg\Delta\varphi \wedge \neg\Delta\neg\varphi$, then $v(\nabla\varphi)$ never reaches 1. Its maximum is only $\frac{3}{4}$.
- Higher-order indeterminate shifts upward: $\nabla\Delta\varphi$, $\nabla\Delta^2\varphi$, etc. peak at increasingly high truth-degrees of φ. Is that the right prediction?
- Arbitrariness of the hedge: why choose x^2 rather than x^3, x^4, or some other function?
- Finite non-collapse, limiting collapse: if $0 < x < 1$, then $x > x^2 > x^4 > x^8 > \cdots$, but $\lim_{n\to\infty} x^{2^n} = 0$.
- Sentence-operator worry: it is clear what "definitely F" means for simple predications, but less clear what Δ means when applied to arbitrary disjunctions, conditionals, or quantified sentences.

Philosophical discussion

- Again, what is graded truth?
- Truth vs. probability. Degrees of truth are not credences, as they compose with connectives differently. If you want beliefs, add a separate probabilistic layer.
- Rate of the decline: linear vs. logistic vs. stepwise drops. Context effects and subject variability. Is it empirically testable in psycholinguistics experiments?
- Higher-order vagueness: degrees avoid a sharp $1/0$ cut, but a crisp "definitely" raises the same issues as the finite approach.

Exercise 1

In K_3^s and K_3^w, the implication $\rightarrow$ is definable using $\vee$ and $\neg$, as $p \rightarrow q \equiv \neg p \vee q$. In 3 we have that $p \rightarrow q \not\equiv \neg p \vee q$.

(i) Furthermore, in 3 we cannot express $p \rightarrow q$ using only $\neg$, $\vee$ and $\wedge$. Prove this fact: that is, prove that for any formula ϕ whose only sentence letters are p and q and has no other connective besides $\neg$, $\vee$ and $\wedge$, there is a valuation v s.t. $v(\phi) \neq v(p \rightarrow q)$.

(ii) Now show that using only $\neg$ and $\rightarrow$, you can define $\wedge$ and $\vee$

Exercise 2

For each $n \in \mathbb{N}_{\geq 2}$, let $_n$ be the n-valued Łukasiewicz logic based on the truth-value set

$$T_n = \left\{ \tfrac{k}{n-1} \;:\; k = 0, 1, \ldots, n-1 \right\} \subseteq [0, 1]$$

with connectives and semantic clauses:

$$v(\neg\varphi) = 1 - v(\varphi)$$
$$v(\varphi \wedge \psi) = \min(v(\varphi), v(\psi))$$
$$v(\varphi \vee \psi) = \max(v(\varphi), v(\psi))$$
$$v(\varphi \to \psi) = \min(1,\, 1 - v(\varphi) + v(\psi))$$

The designated value is 1. For $\Gamma \subseteq$ Form and formula φ,

$\Gamma \models_n \varphi$ iff for every n-valuation v, if $v(\gamma) = 1$ for all $\gamma \in \Gamma$, then $v(\varphi)$

Prove the following fact, for $k, l \in \mathbb{N}_{\geq 2}$:

$$\Gamma \models_l \phi \;\Rightarrow\; \Gamma \models_k \phi \quad \text{iff} \quad (k-1) \text{ divides } (l-1)$$

Exercise 3

Consider the Weak Kleene K_3^w three-valued logic. For finite (possibly empty) sets of formulas Γ, Δ, we write $\Gamma \models_{K_3^w} \Delta$ iff for every K_3^w-valuation v, if $v(\gamma) = 1$ for all $\gamma \in \Gamma$, then $v(\delta) = 1$ for some $\delta \in \Delta$. We define $\Gamma \models_{CL} \Delta$ analogously, using classical valuations.

(a) Show that for all finite sets of formulas Γ, Δ:
$\Gamma \models_{K_3^w} \Delta$ **iff** there is a subset $\Delta' \subseteq \Delta$ such that
  (i) $\Gamma \models_{CL} \Delta'$, and
  (ii) every sentence letter occurring in a formula in Δ' occurs in some formula in Γ.

(b) Show that the corresponding statement for single *non-empty* conclusions fails. In particular, show that there are formulas ϕ, ψ such that $\phi \models_{K_3^w} \psi$, but it is not the case that both
  (i) $\phi \models_{CL} \psi$, and
  (ii) every sentence letter occurring in ψ occurs in ϕ.