Lecture 3: {Super/Sub}valuationism

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Formal Theories of Vagueness
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Readings

Suggested:

- Cobreros, Pablo & Tranchini, Luca (2019). *Supervaluationism, Subvaluationism and the Sorites Paradox*. In Sergi Oms & Elia Zardini (eds.), *The Sorites Paradox*. New York, NY: Cambridge University Press. pp. 38–62.
- Fine, Kit (1975). “Vagueness, Truth and Logic.” *Synthese* 30, no. 3–4. pp. 265–300.

Outline

Making things precise

Bas van Fraassen

Kit Fine

- **Supervaluationism** (van Fraassen 1966; Fine 1975): evaluate formulas over a range of *admissible precisifications*.
- A *precisification* is a classical "sharpening" of the vocabulary that preserves clear positives and clear negatives.
- Example: the predicate *heap* may be sharpened to "has at least n grains of sand," for many choices of n (e.g. $n = 1000, 1001, \ldots$).
- Thus, multiple precisifications are admissible: supervaluationism does not single out a *unique* cutoff.

A simple example

- Suppose there are just two indeterminate cases:
  - Bob, shorter than John
  - John, slightly taller than Bob
- There are three **admissible** precisifications of *tall*:
  - Bob and John are both tall.
  - Bob and John are both not tall.
  - John is tall and Bob is not tall.
- One precisification is **not admissible**:
  - Bob is tall, but John is not tall.
- Acceptable precisifications must preserve the relevant **penumbral connection**:

$$\mathrm{Tall}(\mathrm{Bob}) \rightarrow \mathrm{Tall}(\mathrm{John})$$

Semantic Indecision

> *The reason it's vague where the outback begins is not that there's this thing, the outback, with imprecise borders; rather there are many things, with different borders, and nobody has been fool enough to try to enforce a choice of one of them as the official referent of the word "outback."*
> *Vagueness is semantic indecision.*
>
> (Lewis 1986: *On the Plurality of Worlds*, p. 213)

[MacFarlane 2016]

Precisification

Definition (Precisification)

Let $v : P \to \{1, i, 0\}$ be a three-valued valuation. We say that a classical valuation v' is a *precisification* of v, and we write $v \leq v'$ iff:

$$v(p) = 1 \Rightarrow v'(p) = 1$$
$$v(p) = 0 \Rightarrow v'(p) = 0$$
$$v(p) = i \Rightarrow v'(p) \in \{0, 1\}$$

	p	q
v	i	0

	p	q
v'_1	1	0
v'_2	0	0

Supertrue and Superfalse

A formula is *supertrue* when it is true on *all* its precisifications; *superfalse* when it is false on all of them.

Definition (Supertruth & Superfalsity)

Let $v : P \to \{1, i, 0\}$ be three-valued and write $\mathrm{Prec}(v) := \{v' : v \leq v'\}$ for its set of classical precisifications. For any formula φ:

$$\text{(Supertruth)} \quad v \models^1 \varphi \iff \forall v' \in \mathrm{Prec}(v) : \; v'(\varphi) = 1$$

$$\text{(Superfalsity)} \quad v \models^0 \varphi \iff \forall v' \in \mathrm{Prec}(v) : \; v'(\varphi) = 0$$

Logical Consequence

We can define both a *global* and a *local* notion of consequence.

Definition (Global consequence)

$\Gamma \models_g \varphi$ iff for all three-valued valuations v, if $v \models^1 \gamma$ for all $\gamma \in \Gamma$, then $v \models^1 \varphi$.

Definition (Local consequence)

$\Gamma \models_l \varphi$ iff for all three-valued valuations v, for all $v' \in \mathrm{Prec}(v)$, if $v'(\gamma) = 1$ for all $\gamma \in \Gamma$, then $v'(\varphi) = 1$.

Global

$\mathrm{Prec}(v)$

if Γ is true everywhere, then φ is true everywhere

Local

$\mathrm{Prec}(v)$

at any selected point: if Γ is true here, then φ is true here

Classical

$\{w\}$

w

the one-point case: if Γ is true at w, then φ is true at w

Local and Global

- Over the base propositional language $\{\neg, \wedge, \vee, \rightarrow\}$, global and local consequence coincide.
- In fact, supervaluationist consequence is equivalent to classical consequence.

Fact (Consequence equivalence)

$$\Gamma \models_g \varphi \text{ iff } \Gamma \models_l \varphi \text{ iff } \Gamma \models_{\mathrm{CL}} \varphi$$

Consequence equivalence

$$\Gamma \models_g \varphi \;\Rightarrow\; \Gamma \models_l \varphi \;\Rightarrow\; \Gamma \models_{\mathrm{CL}} \varphi \;\Rightarrow\; \Gamma \models_g \varphi$$

Global	**Local**	**Classical**
$\mathrm{Prec}(v)$	$\mathrm{Prec}(v)$	$\{w\}$
		w
if Γ is true everywhere, then φ is true everywhere	at any selected point: if Γ is true here, then φ is true here	the one-point case: if Γ is true at w, then φ is true at w

- **Global implies local:** A classical precisification is fully precise. So if the premises are true at that precisification, global validity guarantees that the conclusion is true there too.
- **Local implies classical:** Every classical valuation is a one-point precisification. Therefore, if local consequence holds, the conclusion must hold under every classical valuation satisfying the premises.
- **Classical implies global:** Every precisification is classical. So if classical consequence holds, then whenever the premises are true at all precisifications, the conclusion is true at all of them as well.

Supervaluations as sets of valuations

- For a three-valued $v : P \to \{1, i, 0\}$, let $\mathrm{Prec}(v)$ be the set of all classical *precisifications* of v.
- Alternatively, start from an arbitrary nonempty set $V \subseteq \{0,1\}^P$ of admissible classical valuations and evaluate formulas pointwise over V.

Supertruth/superfalsity lift as:

$$V \models^1 \varphi \iff \forall v' \in V :\ v'(\varphi) = 1$$

$$V \models^0 \varphi \iff \forall v' \in V :\ v'(\varphi) = 0$$

This is equivalent to the original definition via v, taking $V = \mathrm{Prec}(v)$

Pointed evaluations

Given a non-empty set of classical valuations V[1], define the pointed satisfaction relation $V, v \models \varphi$ for $v \in V$ by:

$V, v \models p$	iff	$v(p) = 1$
$V, v \models \neg\varphi$	iff	$V, v \not\models \varphi$
$V, v \models \varphi \wedge \psi$	iff	$V, v \models \varphi$ and $V, v \models \psi$
$V, v \models \varphi \vee \psi$	iff	$V, v \models \varphi$ or $V, v \models \psi$
$V, v \models \varphi \to \psi$	iff	$V, v \not\models \varphi$ or $V, v \models \psi$

Definition (Supertruth)

Given a non-empty V, a formula φ is supertrue iff $V, v \models \varphi$ for all $v \in V$. We write $V \models^1 \varphi$.

Likewise, φ is superfalse iff $V \models^1 \neg\varphi$. We write $V \models^0 \varphi$.

[1] Allowing empty V does not change the resulting logic (as $\varnothing \models^1 \varphi$ for any φ), but it matters for satisfiability.

Local and Global

Likewise, global and local consequence can be recast in this way:

Definition (Global consequence)

$\Gamma \models_g \varphi$ iff for all non-empty V, if $V \models^1 \gamma$ for all $\gamma \in \Gamma$, then $V \models^1 \varphi$.

Definition (Local consequence)

$\Gamma \models_l \varphi$ iff for all non-empty V, for all $v \in V$, if $V, v \models \gamma$ for all $\gamma \in \Gamma$, then $V, v \models \varphi$.

Bivalence vs. Law of Excluded Middle

- **Failure of bivalence:** There are non-empty V and p with neither $V \models^1 p$ nor $V \models^1 \neg p$.
- **Validity of LEM:** Every classical tautology is supertrue. In particular for every non-empty V, $V \models^1 \; p \vee \neg p$.

Modelling the Sorites

- Recall the descending Sorites sequence from p_1 to p_N.

$$\begin{array}{l} p_1 \\ p_1 \to p_2 \\ \vdots \\ p_{N-1} \to p_N \\ \hline p_N \end{array}$$

- A faithful model V for a descending Sorites series satisfies:
  - $V \models^1 p_1$
  - $V \models^0 p_N$
  - $\exists k \ (1 < k < N)$ with $V \not\models^1 p_k$ and $V \not\models^1 \neg p_k$
  - $\forall v \in V \ \forall m \in \{1, \ldots, N\}$:
    $$\begin{cases} v(p_m) = 1 \Rightarrow \forall k \ (1 \leq k \leq m \Rightarrow v(p_k) = 1) \\ v(p_m) = 0 \Rightarrow \forall k \ (m \leq k \leq N \Rightarrow v(p_k) = 0) \end{cases}$$

An example

- Take $N = 5$. A faithful model is:

V	p_1	p_2	p_3	p_4	p_5
v_1	1	0	0	0	0
v_2	1	1	0	0	0
v_3	1	1	1	0	0
v_4	1	1	1	1	0

$V \models^1 p_1$

$V \not\models^1 (p_1 \to p_2)$ and $V \not\models^0 (p_1 \to p_2)$

$V \not\models^1 (p_2 \to p_3)$ and $V \not\models^0 (p_2 \to p_3)$

$V \not\models^1 (p_3 \to p_4)$ and $V \not\models^0 (p_3 \to p_4)$

$V \not\models^1 (p_4 \to p_5)$ and $V \not\models^0 (p_4 \to p_5)$

$V \models^0 p_5$

An example (cut-off as a disjunction)

V	p_1	p_2	p_3	p_4	p_5
v_1	1	0	0	0	0
v_2	1	1	0	0	0
v_3	1	1	1	0	0
v_4	1	1	1	1	0

- $A : (p_1 \to p_2) \wedge (p_2 \to p_3) \wedge (p_3 \to p_4) \wedge (p_4 \to p_5)$
- $\neg A : (p_1 \wedge \neg p_2) \vee (p_2 \wedge \neg p_3) \vee (p_3 \wedge \neg p_4) \vee (p_4 \wedge \neg p_5)$
- A is superfalse, and $\neg A$ is supertrue. However, none of the disjuncts is supertrue (no specific cut-off point).

Assessing the situation

Supervaluationist answer to the Sorites: not all conditional premises are supertrue (so the argument is blocked), *without committing to which step* (no conditional is superfalse).

The notion of truth

- Supervaluationism lifts truth from a single valuation to a set of valuations. This echoes two frameworks:
    1. **Modal logic:** formulas are evaluated relative to possible worlds. A formula is true in a model if it is true in **all worlds of the model**.
    2. **Team semantics:** formulas are evaluated w.r.t. a *team* (set of valuations). A formula is true in a model if it is true in **all valuations of the team**.
- Over the base language (without modal operators), both are equivalent to classical logic.
- But lifting truth to sets of valuations yields loss of bivalence.
- Adding modal operators (next) or defining different connectives yields a logic whose consequence is different from classical propositional logic.

Outline

Definitely operator

- Add a determinacy operator Δ ("definitely"). Think of Δ as a necessity operator where precisifications act as worlds.
- Intuitively, Δp is true at a precisification v' of v iff p holds at *all* precisifications of v.
- For simplicity, first take the accessibility relation to be universal:

$$V, v \models \Delta\varphi \quad \text{iff} \quad \forall v' \in V :\ V, v' \models \varphi$$

Deduction Theorem

$$p \models_g \Delta p$$

$$\not\models_g \; p \to \Delta p$$

So the deduction theorem fails: $\Gamma, \varphi \models_g \psi \;\not\Rightarrow\; \Gamma \models_g \; \varphi \to \psi$

Global vs. Local with Δ

$$\varphi \models_g \Delta\varphi$$

$$\varphi \not\models_l \Delta\varphi$$

Semantics

- Formulas are evaluated not just wrt a set of valuations but a pair $M = \langle V, R\rangle$ with $V \neq \varnothing$ and $R \subseteq V \times V$. Each $v \in V$ is a classical valuation $v : P \to \{0, 1\}$.

$$
\begin{array}{lll}
M, v \models p & \text{iff} & v(p) = 1 \\
M, v \models \neg\varphi & \text{iff} & M, v \not\models \varphi \\
M, v \models \varphi \wedge \psi & \text{iff} & M, v \models \varphi \text{ and } M, v \models \psi \\
M, v \models \varphi \vee \psi & \text{iff} & M, v \models \varphi \text{ or } M, v \models \psi \\
M, v \models \varphi \to \psi & \text{iff} & M, v \not\models \varphi \text{ or } M, v \models \psi \\
M, v \models \Delta\varphi & \text{iff} & \forall v' \in V \; (\, vRv' \Rightarrow M, v' \models \varphi \,)
\end{array}
$$

Supertruth: $M \models^1 \varphi :\Longleftrightarrow \forall v \in V \; (M, v \models \varphi)$

Global: $\Gamma \models_g \varphi \iff \forall M \; (M \models^1 \Gamma \Rightarrow M \models^1 \varphi)$

Local (modal logic): $\Gamma \models_l \varphi \iff \forall M \, \forall v \in V (M, v \models \Gamma \Rightarrow M, v \models \varphi)$

Outline

Higher-order Vagueness

- Define indeterminate: $\nabla\varphi := \neg\Delta\varphi \wedge \neg\Delta\neg\varphi$.
- With *universal* accessibility: $\nabla\,\Delta p$ is not satisfiable.

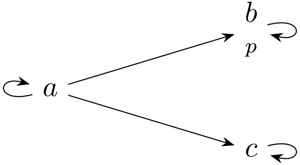
Frame constraints $\leftrightarrow$ modal axioms (for Δ)

$\mathbf{T}$ (reflexive)	$\Delta\varphi \to \varphi$
$\mathbf{4}$ (transitive)	$\Delta\varphi \to \Delta\Delta\varphi$
$\mathbf{B}$ (symmetric)	$\varphi \to \Delta\neg\Delta\neg\varphi$
$\mathbf{5}$ (euclidean)	$\neg\Delta\varphi \to \Delta\neg\Delta\varphi$
$\mathbf{S5}$ / equivalence	T+4+5; T+4+B; T+B+5 $\Rightarrow \Delta\Delta\varphi \vee \Delta\neg\Delta\varphi$ is valid.

- **To allow higher-order vagueness:** we need to drop some axioms. Which ones to keep for a 'Definitely' operator?

R is reflexive and transitive (S4-frames)

In reflexive and transitive frames, higher-order vagueness for Δ is *possible*: $\nabla\Delta p$ is satisfiable.



- $R[b] = \{b\}$, so $M, b \models \Delta p$.
- $R[c] = \{c\}$, so $M, c \not\models \Delta p$.
- Therefore $M, a \models \nabla\Delta p$.

Outline

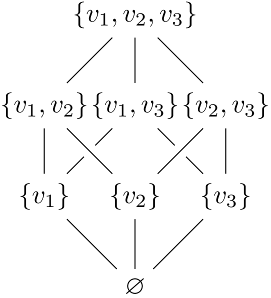
Supervaluations and truth-functionality

- **Truth-functionality:** the truth of a complex sentence is a function of the truth of its constituents.
- Supervaluationist theories are not truth-functional at the level of supertruth/superfalsity.
- For instance, the supertruth of $\neg p$ is not determined solely by whether p is supertrue.

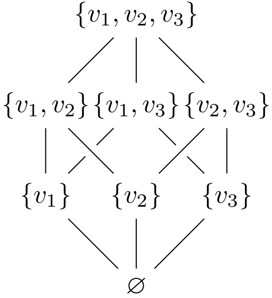
[Cobreros and Tranchini 2014; Field 2008]

An algebraic perspective

- Given V, the powerset $\mathcal{P}(V)$ identifies formulas with their set of supporting valuations.
- Take $V = \{v_1, v_2, v_3\}$ with $v_1(p) = 1$, $v_2(q) = 1$, others 0. Then p corresponds to $\{v_1\}$, and $p \vee q$ to $\{v_1, v_2\}$.



Functionality via sets

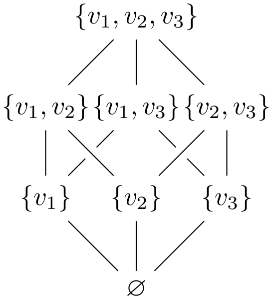


Functionality is preserved *extensionally*:

$$
\begin{array}{lcl}
f(p) & = & \{v \in V :\ v(p) = 1\} \\
f(\neg\varphi) & = & V \setminus f(\varphi) \\
f(\varphi \vee \psi) & = & f(\varphi) \cup f(\psi) \\
f(\varphi \wedge \psi) & = & f(\varphi) \cap f(\psi)
\end{array}
$$

φ is supertrue in V iff $f(\varphi) = V$; superfalse iff $f(\varphi) = \varnothing$.

Supervaluations and degrees



- This representation suggests a degree-theoretic flavor.
- However, take $V = \{v_1, v_2, v_3\}$ with $v_1(p) = 1$, $v_2(q) = 1$, others 0.
- Then $f(p) = \{v_1\}$ and $f(\neg p) = \{v_2, v_3\}$. But supervaluationism does *not* rank p as 'less true' than $\neg p$.

Discussion

- Disjunction can be supertrue without any supertrue disjunct, undermining the intuitive idea that a true disjunction is always made true by one of its disjuncts.
- While theoremhood over the base language is classical, familiar *argument forms* needn't be globally supervalid with Δ (e.g., conditional proof)
- Higher-order vagueness is problematic with S5 axioms
- Supervaluationists say "there is an n where the cutoff occurs" (since $\exists n\,(\varphi(n) \wedge \neg\varphi(n{+}1))$ is supertrue) yet deny there *really is* a sharp cutoff. Some see this as talking *as if* sharp boundaries exist.

Outline

Supertrue vs Subtrue

- **Supervaluationism** takes a sentence to be true just in case it is true in **all** of its possible precisifications.
- **Subvaluationism** takes a sentence to be true just in case it is true in **some** of its possible precisifications.
- Supervaluationism: vagueness as *underdetermination*. Indeterminate cases are neither supertrue nor superfalse.
- Subvaluationism: vagueness as *overdetermination*. Indeterminate cases are both subtrue and subfalse.

Subtrue and Subfalse

Definition (Subtruth & Subfalsity)

Let $V \neq \varnothing$ be a set of classical valuations. Using the relation $V, v \models \varphi$ from before, we define for any formula φ:

$$\textbf{(Subtruth)} \quad V \models^{\exists 1} \varphi \iff \exists v \in V : V, v \models \varphi$$

$$\textbf{(Subfalsity)} \quad V \models^{\exists 0} \varphi \iff \exists v \in V : V, v \not\models \varphi$$

Global and Local consequence

Definition (Global subvaluationist consequence)

$\Gamma \models^{\exists}_{g} \varphi$ iff for all non-empty V, if $V \models^{\exists 1} \gamma$ for all $\gamma \in \Gamma$, then $V \models^{\exists 1} \varphi$.

Definition ('Local' subvaluationist consequence)

$\Gamma \models^{\exists}_{l} \varphi$ iff for all non-empty V, if for all $\gamma \in \Gamma$ there exists $v \in V$ with $V, v \models \gamma$, then there exists $v' \in V$ with $V, v' \models \varphi$.

Here the collapse holds unrestrictedly, as they are equivalent by definition.

Yet another logical consequence

Definition (Another consequence)

$\Gamma \models^{\exists}_{\forall} \varphi$ iff for all non-empty V, if there exists $v \in V$ with $V, v \models \gamma$ for all $\gamma \in \Gamma$, then there exists $v' \in V$ with $V, v' \models \varphi$.

$$\Gamma \models^{\exists}_{\forall} \varphi \iff \Gamma \models_{CL} \varphi$$

The latter holds over the base language. For the $(\Rightarrow)$ direction take a singleton $V = \{v\}$.

Global consequence

- Global subvaluationism and classical consequence do not coincide:

$$\Gamma \models_g^\exists \varphi \Rightarrow \Gamma \models_{CL} \varphi$$

$$\Gamma \models_g^\exists \varphi \not\Leftarrow \Gamma \models_{CL} \varphi$$

- Global subvaluationism is paraconsistent:

$$\{p, \neg p\} \not\models_g^\exists q$$

- Note: consequence is different from LP:
  $p \wedge \neg p \models_g^\exists q$ and $\{p, q\} \not\models_g^\exists p \wedge q$

Sets of Conclusions

- So far we worked with single-conclusion consequence $\Gamma \models \varphi$
- What about a **set of conclusions** $\Gamma \models \Phi$?

$\Gamma \models \Phi \Longleftrightarrow \neg\exists\, v\ (v \models \Gamma$ and $v \models \neg\Phi)$, where $\neg\Phi := \{\neg\varphi : \varphi \in \Phi\}$

when all premises hold, at least one member of Φ must hold (no joint countermodel).

Global sup/sub-valuationist consequence are the dual of each other:

$$\Gamma \models_g \Phi \iff \neg\Phi \models_g^{\exists} \neg\Gamma$$

Bivalence fails in sup.s: $\not\models_g \{\varphi, \neg\varphi\}$
Split-inconsistency fails in sub.s: $\{\varphi, \neg\varphi\} \not\models_g^{\exists} \varnothing$

The Sorites

V	$\varphi(1)$	$\varphi(2)$	$\varphi(3)$	$\varphi(4)$
v_1	1	0	0	0
v_2	1	1	0	0
v_3	1	1	1	0

Conditionals $C_k : \ \varphi(k) \to \varphi(k{+}1)$*:* each C_k is true at all $v \neq v_k$ and false at v_k.

$$\forall k : \ V \models^{\exists 1} C_k \quad \text{but} \quad V \not\models^{\exists 1} \bigwedge_k C_k$$

- Hence every step holds somewhere (it is subtrue), but not all together at one v.
- But the *argument is invalid* since **Modus Ponens can fail** for a conditional that is both subtrue and subfalse

MP is not valid under SbV

Let $V = \{v_1, v_2\}$ with $v_1(p) = 1,\ v_1(q) = 0$ and $v_2(p) = 0,\ v_2(q) = 0$.

$$\underbrace{V \models^{\exists 1} p}_{v_1} \quad \underbrace{V \models^{\exists 1} (p \to q)}_{v_2} \quad \underbrace{V \not\models^{\exists 1} q}_{\text{no } v}$$

So $p,\ (p \to q)\ \not\models^{\exists}_{g} q$.

Each conditional C_k is subtrue (witness $v \neq v_k$) and $\varphi(1)$ is subtrue, but $\varphi(N)$ need not be subtrue. The chain of MP steps can break at the valuation where some C_k is false.

Different forms and asymmetries

- Consider a single line form of the argument: $\big(p_1 \wedge (p_1 \to p_2) \wedge (p_2 \to p_3) \wedge \ldots\big) \Rightarrow p_n$
- Then the argument is valid for subvaluationism.
- But there is no *single* v which makes all the steps true and hence the universal $\forall n\,(p_{n-1} \to p_n)$ is **not even subtrue.**
- Thus the Sorites is here blocked, even though the argument is valid.
- Subvaluationism is thus committed to different answers, depending on the form of the paradox.
- To be fair, supervaluationism also displays this asymmetry for sets of conclusions:

Supervaluationism ($\models_g$)

$$p \vee q \;\not\models_g\; \{p, q\}$$
$$\{p, q\} \;\models_g\; \{p, q\}$$

Subvaluationism ($\models_g^{\exists}$)

$$\{p, q\} \;\not\models_g^{\exists}\; p \wedge q$$
$$\{p, q\} \;\models_g^{\exists}\; \{p, q\}$$

Outline

Setup: higher-order Sorites

- Let T_k mean: item k in the Sorites series is tall.
- Assume clear endpoints: T_1 and $\neg T_m$.
- As usual, the operator Δ means *definitely*. Iterated applications express higher-order determinacy, as in $\Delta T_k, \Delta^2 T_k, \Delta^3 T_k, \ldots$
- First step: if k is definitely tall, then $k+1$ should not already be definitely not tall.
- More generally: $\Delta^n T_k$ should not sharply give way to $\neg\Delta^n T_{k+1}$.
- The truth-gap theorist wants the gap principles: $\mathrm{GP}_n(k):\ \Delta\Delta^n T_k \to \neg\Delta\neg\Delta^n T_{k+1}$
- Equivalently, in contrapositive form: $\Delta\neg\Delta^n T_{k+1} \to \neg\Delta^{n+1} T_k$
- Fara's argument shows that these gap principles, together with global Δ-introduction, generate inconsistency.

[Fara 2003; Cobreros 2008, 2011]

The Fara ladder

Assume T_1 and $\neg T_m$, and use the rule $\varphi \Rightarrow \Delta\varphi$, which is valid for global supervaluationist consequence.

- From the negative endpoint: $\neg T_m \Rightarrow \Delta\neg T_m$
- By the first gap principle: $\Delta\neg T_m \Rightarrow \neg\Delta T_{m-1}$
- Apply Δ-intro again: $\neg\Delta T_{m-1} \Rightarrow \Delta\neg\Delta T_{m-1}$
- By the next gap principle: $\Delta\neg\Delta T_{m-1} \Rightarrow \neg\Delta^2 T_{m-2}$
- Iterating this pattern yields $\neg\Delta^{m-1} T_1$

But from the positive endpoint, repeated Δ-introduction gives

$T_1 \Rightarrow \Delta T_1 \Rightarrow \cdots \Rightarrow \Delta^{m-1} T_1$

$\Delta^{m-1} T_1$ and $\neg\Delta^{m-1} T_1$

Global truth is too external

Global consequence preserves truth at *all* points of the model: $\Gamma \models_g \varphi$ iff, whenever every $\gamma \in \Gamma$ holds everywhere, φ also holds everywhere.

- This makes Δ-introduction valid: $\varphi \models_g \Delta\varphi$.
- But it also makes the relevant notion of truth insensitive to the current precisification.
- If φ is globally true at one point, it is globally true at every point.
- So there can be no indeterminate cases of *global truth* itself.

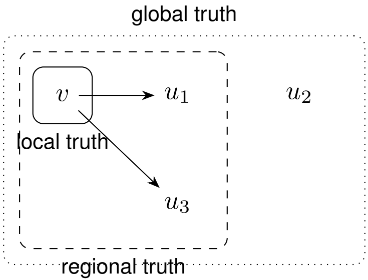
Regional truth

Cobreros's idea: preserve truth not at a single point, and not at all points, but throughout the *region* admitted by the current point.

Definition (Regional truth)

Let $M = \langle V, R\rangle$ be a model. A formula φ is *regionally true at* v iff, for every $u \in V$, if vRu, then $M, u \models \varphi$.

Equivalently, $M, v \models^r \varphi$ iff $M, v \models \Delta\varphi$.



Regional consequence

Definition (Regional consequence)

$\Gamma \models_r \varphi$ iff, for every model $M = \langle V, R \rangle$ and every $v \in V$, if every $\gamma \in \Gamma$ is regionally true at v, then φ is regionally true at v.

Using $\Delta\Gamma := \{\Delta\gamma : \gamma \in \Gamma\}$, this is equivalent to $\Gamma \models_r \varphi$ iff $\Delta\Gamma \models_l \Delta\varphi$.

With reflexive but not necessarily transitive admissibility, $\models_l \subsetneq \models_r \subsetneq \models_g$.

Why Δ-introduction fails regionally

Assume reflexive accessibility, with w_0Rw_1 and w_1Rw_2, but not w_0Rw_2.

$$\begin{tikzcd} w_0 :\ p \arrow[loop above] \arrow[r] & w_1 :\ p,\ \neg\Delta p \arrow[loop above] \arrow[r] & w_2 :\ \neg p \arrow[loop above] \end{tikzcd}$$

- p is regionally true at w_0: it holds throughout the w_0-region.
- But Δp is not regionally true at w_0.

So $p \not\models_r \Delta p$

The final picture

- Local validity is too weak for a truth-gap theory: truth at each individual precisification is bivalent.
- Global validity is too strong: it validates $\varphi \Rightarrow \Delta\varphi$, and global truth cannot vary between precisifications.
- Regional validity preserves truth throughout the precisifications admissible from the current standpoint.
- What counts as admissible may therefore vary from one precisification to another.
- This variation allows truth itself to have indeterminate cases, thereby providing a model of higher-order vagueness.

Exercises

1. Show that:

$$\begin{array}{lll}
\text{(a) } \Gamma \models_l \varphi & \Rightarrow & \Gamma \models_g \varphi \\
\text{(b) } \Gamma, \varphi \models_l \psi & \Leftrightarrow & \Gamma \models_l \varphi \rightarrow \psi \\
\text{(c) } \Gamma \models_g \varphi \rightarrow \psi & \Rightarrow & \Gamma, \varphi \models_g \psi \\
\text{(d) } \models_g \varphi & \Leftrightarrow & \models_l \varphi \\
\text{(e) } \varphi \models_g \psi & \not\Rightarrow & \neg\psi \models_g \neg\varphi
\end{array}$$

   (f) If R is not reflexive, then $\not\models (\Delta\varphi \rightarrow \varphi)$

   (g) $\nabla\Delta p$ is satisfiable on reflexive and symmetric frames

2. On the set-theoretic preservation of functionality, add the clauses for:
   2.1 Universal Δ
   2.2 General Δ