Lecture 5: ST-logic & Higher order vagueness

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Formal Theories of Vagueness
ESSLLI 2026, Prague

7 August 2026

Tolerance and Vagueness

- Measurement Tolerance: $\forall x, y : (x \sim_P y \supset (x \not>_P y \wedge y \not>_P x)$

- Kamp (1981), Wright, Dummett:

  What makes a predicate vague, is that it is tolerant

- Tolerance principle: **[P]** $\forall x, y : (Px \wedge x \sim_P y) \supset Py$

- Tolerance is **constitutive** of its meaning.

Solutions to the Sorites

All: give up on $[\mathbf{P}]$ $\quad \forall x, y \in I : (P(x) \wedge x \sim y) \supset P(y)$

- Supervaluation: $[\mathbf{P}]$ false,
  but no particular instantiation (super)true.
- Other solutions: *weakening* of $[\mathbf{P}]$ is true.
  1. Fuzzy logic: $[\mathbf{P}]$ *almost* true
  2. Three-valued logic
  3. Epistemics
  4. Contextual solution
  5. Pragmatic solution
- None of them takes tolerance seriously (enough)!

Tolerant interpretation

Start with FOL model, with $\sim_P$ a non-transitive similarity relation

- $M \models^c \phi$ defined in the usual way.
- $\boxed{M \models^t P(\underline{a})}$ iff $\boxed{\exists d \sim_P a}$: $\boxed{M \models^c P(\underline{d})}$ with $\underline{d}$ name for d
- $M \models^t \neg\phi$ iff $M \not\models^t \phi$
- $M \models^t \phi \wedge \psi$ iff $M \models^t \phi$ and $M \models^t \psi$
- $M \models^t \forall x\phi$ iff for all $d \in I_M$: $M \models^t \phi[^x/_{\underline{d}}]$.
- $[[P\underline{a}]]^c \subset [[P\underline{a}]]^t$ $\leadsto$ Tolerant truth **weaker**
- $[[\neg P\underline{a}]]^c \supset [[\neg P\underline{a}]]^t$ $\leadsto$ Tolerant truth **stronger**

Satisfaction: TCS (JoPL)

- $M \models^t P(\underline{a})$ iff $\exists d \sim_P a : M \models^c P(\underline{d})$
- $M \models^t \neg\phi$ iff $M \not\models^s \phi$
- $M \models^t \phi \wedge \psi$ iff $M \models^t \phi$ and $M \models^t \psi$
- $M \models^t \forall x\phi$ iff for all $d \in D : M \models^t \phi[\underline{d}/x]$

- $M \models^s P(\underline{a})$ iff $\forall d \sim_P a : M \models^c P(\underline{d})$
- $M \models^s \neg\phi$ iff $M \not\models^t \phi$
- $M \models^s \phi \wedge \psi$ iff $M \models^s \phi$ and $M \models^s \psi$
- $M \models^s \forall x\phi$ iff for all $d \in D : M \models^s \phi[\underline{d}/x]$

- $\Gamma \models^{st} \psi$ iff $\forall M, \forall \phi \in \Gamma : M \models^s \phi \Rightarrow M \models^t \psi$

Properties

- $[[\phi]]^s \subseteq [[\phi]] \subseteq [[\phi]]^t \quad \rightsquigarrow \quad$ Also for $\neg Pa$
- $Pa \wedge \neg Pa$ not a tolerant contradiction
- $Pa \vee \neg Pa$ not a strict tautology
- Connection with LP and K3?

Three-valued

A ST-model is a three-valued model M, of the sort familiar from Strong Kleene logic or LP. The domain must include $\mathcal{L}$.

Where I_M is an interpretation:

- $I_M(A) \in \{1, \frac{1}{2}, 0\}$
- $I_M(\top) = 1; I(\bot) = 0$
- $I_M(A \wedge B) = \min(I_M(A), I_M(B))$
- $I_M(\neg A) = 1 - I_M(A)$
- $I_M(\forall x A) = \min(\{I'_M(A) : I'$ is an x-variant of $I_M\})$

- $\Gamma \models^{st} \psi$ iff $\forall M, \forall \phi \in \Gamma : I_M(\phi) = 1 \Rightarrow I_M(\psi) \neq 0$

st's classicality

3-valued Logics tend to lose certain familiar classical inferences.

But st features all of:

- Excluded middle: $\top \models^{st} A \vee \neg A$
- Explosion: $A \wedge \neg A \models^{st} \bot$
- Modus ponens: $A, A \supset B \models^{st} B$
- Identity: $\models^{st} A \supset A$
- Contraction: $A \supset (A \supset B) \models^{st} A \supset B$

Excluded middle:	$\top$	$\models^{st}$	$A \vee \neg A$
Explosion:	$A \wedge \neg A$	$\models^{st}$	$\bot$
Modus ponens:	$A, A \supset B$	$\models^{st}$	B
Identity:	$\top$	$\models^{st}$	$A \supset A$
Contraction:	$A \supset (A \supset B)$	$\models^{st}$	$A \supset B$

st's classicality (cont)

st also preserves some important classical meta-inferences:

- Monotonicity
- Reasoning by cases
- $\Gamma, A \models^{st} \Delta$ iff $\Gamma \models^{st} \neg A, \Delta$ (and with $A, \neg A$ reversed).
- $\Gamma \models^{st} \Delta$ iff $\Gamma' \models^{st} \Delta'$, where Γ' comes from Γ by possibly conjoining some members, and Δ' from Δ by possibly disjoining.
- Deduction theorem:
  $\Gamma, A \models^{st} B, \Delta$ iff $\Gamma \models^{st} A \supset B, \Delta$.

Nontransitivity

st is, however, nontransitive.

Example

Take a sentence λ that takes value $\frac{1}{2}$ on any ST-model. Thus,

- $\top \models^{st} \lambda$, and
- $\lambda \models^{st} \bot$.

Nevertheless,

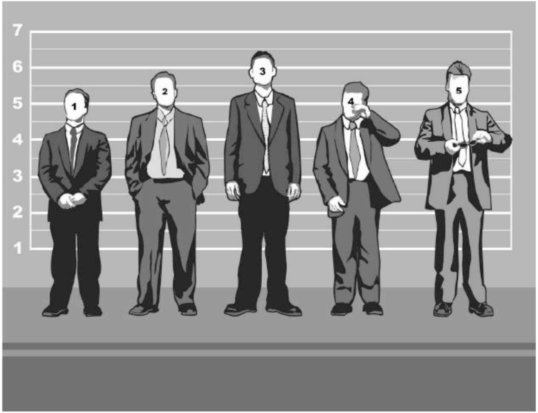
- $\top \not\models^{st} \bot$.

Its nontransitivity blocks triviality.

Pragmatic analysis of Penumbral Connections

- Fine: Global quantification crucial for Penumbral connections
- Indeed, our analysis non-standard for $Pa \wedge \neg Pa \quad Pa \vee \neg Pa$
- But we don't see why this is problematic (see below)
- But if a slightly shorter than $b \Rightarrow M \models^t Ta \wedge \neg Tb$ is possible!!
- Pragmatics: always interpret as strongly as possible $\Rightarrow$ strictly
- Strictly interpretation of $Pa \wedge \neg Pb \Rightarrow a >_P b$
  But this is impossible if a slightly shorter than b

Use of contradictions (Alxatib & Pelletier, Ripley)



- V: Is 2 tall? A: no. V: Is 2 not tall? A: no.
- V: is 2 tall and not tall? A: yes.

Experiment 1: Ripley

Contradictions for borderline cases:

1. The circle is near the square and isn't near the square
2. The circle both is and isn't near the square
3. The circle neither is near the square nor isn't near the square
4. The circle neither is nor isn't near the square

All agreed with to same amount

Contradictions and excluded middle

Data from Ripley (2009), Alxatib and Pelletier (2009), Serchuk, Hargreaves and Zach (2008) indicate that:

- Subjects are more willing to accept $P(x) \wedge \neg P(x)$ for borderline cases (Ripley 2009, Alxatib and Pelletier 2009)
- Subjects are less willing to accept $P(x) \vee \neg P(x)$ for borderline cases (Serchuk et al. 2008)

The data agree with the strongest meaning hypothesis.

Problem: 'John is tall and not tall, **or** Mary is rich'

Solution (Cobreros et al, 2015) for this made use of truth-makers.

ST and Evans' problem of Vague Identity

Assume $I(\nabla(\underline{a} = \underline{b})) = 1$ iff $I(\underline{a} = \underline{b}) = \frac{1}{2}$

1. $\nabla(\underline{a} = \underline{b})$
2. $\lambda x(\nabla \underline{a} = x)\underline{b}$ — Lambda abstraction
3. $\neg\lambda x(\nabla \underline{a} = x)\underline{a}$
4. $\underline{a} \neq \underline{b}$ — Leibniz' Law
5. $\neg\nabla(\underline{a} = \underline{b})$

- all steps are ST-valid
- $2, 3 \models^{st} 4$ $4 \models^{st} 5,$ because $I(\underline{a} \neq \underline{b}) = \frac{1}{2}$
- but $1 \not\models^{st} 5$

Truth, and the liar paradox

$\lambda = \neg T\langle\lambda\rangle =$ 'This sentence is not true'

$$\begin{array}{c}
\top \\
T\lambda \vee \neg T\lambda \\
\\
\begin{array}{c|c}
T\lambda & \neg T\lambda \\
\lambda & \lambda \\
\neg T\lambda & T\lambda \\
T\lambda \wedge \neg T\lambda & T\lambda \wedge \neg T\lambda
\end{array} \\
\\
T\lambda \wedge \neg T\lambda \\
\bot
\end{array}$$

Truth and the Curry paradox

$\kappa = T\langle\kappa\rangle \supset \bot =$ 'If this sentence is true, then $\bot$'

$$
\begin{array}{l}
\top \\
\quad | \; T\langle\kappa\rangle \\
\quad | \; \kappa \\
\quad | \; T\langle\kappa\rangle \supset \bot \\
\quad | \; \bot \\
T\langle\kappa\rangle \supset \bot \\
\kappa \\
T\langle\kappa\rangle \\
\bot
\end{array}
$$

Language

$\mathcal{L}$ is a first-order language with $\top$, $\bot$, a distinguished name $\langle A \rangle$ for every formula A, and truth predicate T. (Can add $=$, but we skip it here.)

$\mathcal{L}$ allows for diagonal formula formation, so we get paradoxical sentences. In particular, there is a sentence λ that is $\neg T\langle \lambda \rangle$, and a sentence κ that is $T\langle \kappa \rangle \supset \bot$.

ST$^+$-models

ST-models treat T like any other predicate.

$I_M(\langle A \rangle) = A$

An ST$^+$-model is an ST-model on which $I(T\langle A \rangle) = I(A)$ for every formula A.

ST$^+$-models are Kripkean fixed points.

Very conservative

Conservative extensions are supported by a Kripkean result: every fixable ST-model can be extended to an ST$^+$-model.

Also, if $\Gamma \vDash^{st} \Delta$, then $\Gamma^\star \vDash^{st}_{+} \Delta^\star$, where * is any uniform substitution.

st^+ is thus a conservative extension of classical logic with transparent truth.

A liar argument

$\lambda = \neg T\langle\lambda\rangle =$ 'This sentence is not true'

$$\top$$
$$T\lambda \vee \neg T\lambda$$

$$\begin{array}{c|c} T\lambda & \neg T\lambda \\ \lambda & \lambda \\ \neg T\lambda & T\lambda \\ T\lambda \wedge \neg T\lambda & T\lambda \wedge \neg T\lambda \end{array}$$

$$T\lambda \wedge \neg T\lambda$$
$$\bot$$

- Every step in the argument is st^+-valid.
- They cannot be chained together.
- LEM only guarantees a tolerantly satisfied conclusion.
- ECQ requires a strictly satisfied premise.

A Curry argument

$\kappa = T\langle\kappa\rangle \supset \bot =$ 'If this sentence is true, then $\bot$'

$$
\begin{array}{l}
\top \\
\quad \left| \begin{array}{l} T\langle\kappa\rangle \\ \kappa \\ T\langle\kappa\rangle \supset \bot \\ \bot \end{array} \right. \\
T\langle\kappa\rangle \supset \bot \\
\kappa \\
T\langle\kappa\rangle \\
\bot
\end{array}
$$

- Again, every step is st^+-valid.
- Again, they cannot be chained together.
- CP only guarantees a tolerantly satisfied conclusion.
- Modus ponens requires strict satisfaction of the antecedent.

Nontransitive solutions

Since $\top \not\models^{st}_{+} \bot$, all formulable paradoxes have non-disastrous solutions. The solutions for more complex paradoxes are like those we've seen: one must keep track of tolerant and strict. Tolerant outputs may not be used where strict inputs are required.

This does not block transitivity in general; transitivity holds, with restrictions, in most usual cases. The blocked reasoning is quite limited.

Further directions

More work done:

- Add a satisfaction predicate, and treat Grelling's paradox
- Extend to attitudes, and treat the Knower paradox
- Add naive comprehension, and explore naive set theory
- Add consequence relation to object-language

- Investigate meta-meta-inferences (lots of work)

- Higher order vagueness and revenge ?

Background: The Problem of Vagueness

- Classical logic is **bivalent**: every sentence is true or false.
- **Vague predicates** (e.g. "is tall", "is bald") resist this:
  - Borderline cases seem *neither* clearly true nor clearly false.
- The **sorites paradox**: a series of small steps leads from a clear truth to a clear falsehood.
- **Higher-order vagueness**: it is also vague *which* objects are borderline.

The Problem: Williamson's Objection

> *"The objection to 2-valued logic was the supposed impossibility of classifying all propositions as true or false; but the phenomenon of second-order vagueness makes it equally hard to classify all vague propositions as true, false or neither. . . . If two values are not enough, three are not enough."*
>
> — Williamson (1994: 111)

Core problem: The same reasoning that motivates moving from 2 to 3 truth values reappears at every level. There is no principled stopping point.

Basic Idea, of first kind of solution

Key Insight

The notion of *admissible precisification* itself has borderline cases. So does the notion of *admissible admissible precisification*, and so on.

Strategy: Extend the 3-valued framework by iterating sets of precisifications across levels.

- Level 0: classical truth values $\{0, 1\}$
- Level 1: sets of classical values (from sets of precisifications)
- Level n: sets of level-$(n-1)$ values

Level 1 Model

A **Level 1 model** is $M_1 = (D, I, P_1)$ where:

- D: domain of individuals
- I: interpretation function for names
- P_1: a set of (level 0) precisifications (total predicate interpretations)

The semantic value of a sentence:

$$M(\phi) = P_1(\phi) = \bigcup_{p \in P_1} \{p(\phi)\}$$

- True in all precisifications $\Rightarrow$ value $\{1\}$
- False in all precisifications $\Rightarrow$ value $\{0\}$
- Borderline $\Rightarrow$ value $\{0, 1\}$ (instead of $\frac{1}{2}$)

Level 1 Example

Language with predicate *Tall*. Let $P_1 = \{p_1, p_2\}$:

$$p_1 = \{d \in D : \text{height}(d) > 1.90\text{m}\}, \quad p_2 = \{d \in D : \text{height}(d) > 1.80\text{m}\}$$

x	Mary	Bob	John
height(x)	1.75m	1.82m	1.92m
$M(\text{Tall}(x))$	$\{0\}$	$\{0, 1\}$	$\{1\}$

Bob is a **borderline case**: tall under p_2, not tall under p_1.

The Value Hierarchy

Truth values are defined recursively:

$$V_0 = \{0,1\}, \qquad V_{k+1} = \wp(V_k) \setminus \{\emptyset\}$$

For each level n, the model is $M = (D, I, P_n)$ with:

$$P_{k+1}(\phi) = \bigcup_{P_k \in P_{k+1}} \{P_k(\phi)\}$$

At level 2, the possible values include:

$\{\{0\}\}$, $\{\{1\}\}$, $\{\{0,1\}\}$, $\{\{0\},\{1\}\}$, $\{\{0\},\{0,1\}\}$, $\{\{1\},\{0,1\}\}$, $\{\{0\},\{1\},\{0,1\}\}$

Level 2 Example

Let $P_2 = \{\{> 1.80, > 1.90\},\ \{> 1.85, > 1.95\}\}$

x	Jo	Bo	Ko	Paul	Pete
height	1.65	1.82	1.87	1.91	1.96
$M(\mathsf{Tall}(x))$	$\{\{0\}\}$	$\{\{0,1\},\{0\}\}$	$\{\{0,1\}\}$	$\{\{1\},\{0,1\}\}$	$\{\{1\}\}$

- Jo: *clearly not tall* at both levels
- Ko: *borderline tall* under both sets of precisifications
- Pete: *clearly tall* at both levels
- Bo, Paul: higher-order borderline cases

Connectives: Recursive Lifting

Logical operations lift from the classical base at each level:

$$u_{k+1} \wedge_{k+1} v_{k+1} = \{u_k \wedge_k v_k : u_k \in u_{k+1},\ v_k \in v_{k+1}\}$$

$$u_{k+1} \vee_{k+1} v_{k+1} = \{u_k \vee_k v_k : u_k \in u_{k+1},\ v_k \in v_{k+1}\}$$

$$\neg_{k+1} u_{k+1} = \{\neg_k u_k : u_k \in u_{k+1}\}$$

Example:

$$M(\mathsf{Tall}(\mathsf{Jo}) \wedge_2 \mathsf{Tall}(\mathsf{Bo})) = \{\{0\}\} \wedge_2 \{\{0,1\},\{0\}\} = \{\{0\}\}$$

Consequence Relations

Define $\Gamma \models_n \phi$: for all models, if all $\gamma \in \Gamma$ have a designated value, so does ϕ.

Choice 1 — 1 occurs somewhere (∃-condition):

$$D_0 = \{1\}, \quad D_{k+1} = \{v \in V_{k+1} \mid \exists u \in v : u \in D_k\}$$

E.g. $D_1 = \{\{1\}, \{0,1\}\}$. Result: $\models_n = \models^{LP}$ for all n.

Choice 2 — 1 occurs everywhere (∀-condition):

$$D_0 = \{1\}, \quad D_{k+1} = \{v \in V_{k+1} \mid \forall u \in v : u \in D_k\}$$

E.g. $D_1 = \{\{1\}\}$. Result: $\models_n = \models^{K3}$ for all n.

Why the Collapse Happens

Key structural observation

At level 2, the three values of V_1 are replicated in their roles:

- $\{\{0\}\}$ behaves like $\{0\}$
- $\{\{1\}\}$ behaves like $\{1\}$
- All other values behave like $\{0, 1\}$

This structural isomorphism generalizes to all higher levels n.

Therefore, the **inferential behavior** at every level mirrors that of a 3-valued system — LP or K3 if one designated value, ST if mixed.

Conclusion

The framework separates two dimensions:

Descriptive power

Increases with each level. More truth values are available to classify borderline, borderline-borderline, etc. statuses.

Inferential behavior

Remains constant. At every level, reasoning collapses back to LP, K3, or ST.

Upshot

Higher order vagueness is **accommodated descriptively** without disrupting the logical core. The hierarchy of values is semantically expressive but inferentially inert.

Extend limitation of expressibility and revenge

Both Strong Kleene (SK) and Weak Kleene (WK) use three truth values:

Value	**Interpretation**
1	True
n	Indeterminate / Borderline
0	False

Key difference: how n interacts with logical connectives.

Strong Kleene vs. Weak Kleene

Strong Kleene (SK)

- n behaves "as much like a truth value as possible"
- $0 \wedge n = 0$ (false wins)
- $1 \vee n = 1$ (true wins)

Weak Kleene (WK)

- n is **infectious**
- $0 \wedge n = n$ (spreads)
- $1 \vee n = n$ (spreads)
- Any formula with an indeterminate component is indeterminate

Weak Kleene Truth Tables

$\wedge$	0	n	1
0	0	n	0
n	n	n	n
1	0	n	1

$\supset$	0	n	1
0	1	n	1
n	n	n	n
1	0	n	1

$\vee$	0	n	1
0	0	n	1
n	n	n	n
1	1	n	1

$\neg$	
0	1
n	n
1	0

Key observation

n contaminates every connective — this is the **infectiousness** property.

The Strict/Tolerant Framework (ST)

Validity is defined via counterexamples:

$\Gamma \models \Delta$ is valid iff no model has all $\gamma \in \Gamma$ valued 1 and all $\delta \in \Delta$ valued 0.

- **sST** $=$ ST logic built on Strong Kleene models
- **wST** $=$ ST logic built on Weak Kleene models

Tolerance Principle

If $P(c_i)$ is true and c_i is indistinguishable from c_{i+1} w.r.t. P, then $P(c_{i+1})$ is tolerantly true — without generating a sorites paradox.

Appealing feature

Both **sST** and **wST** preserve classical logic

Modelling Vagueness in ST

Setup for a sorites series:

1. $v(P(c_0)) = 1$ (clear positive case)
2. $\exists\, c_j :\; v(P(c_j)) = n$ (borderline cases exist)
3. $\exists\, c_r :\; v(P(c_r)) = 0$ (clear negative case)
4. $v(c_i \sim_P c_{i+1}) = 1$ for all i (indistinguishability)
5. Truth values decrease monotonically: $0 \leq n \leq 1$

ST validates the tolerance principle without collapsing into contradiction.

because $\models$ is non-transitive, or **Cut** doesn't hold

Extend expressibility: The Groundedness Predicate

To handle higher-order vagueness, introduce a **groundedness predicate** Grd:

$$v(Grd(\ulcorner \varphi \urcorner)) = \begin{cases} 1 & \text{if } v(\varphi) \in \{0, 1\} \\ 0 & \text{if } v(\varphi) = n \end{cases}$$

- Grd is **bivalent**: every sentence is either grounded or not.
- $Grd(\ulcorner P(c) \urcorner)$ says: "it is *determinate* whether c is P".
- Captures the idea of **determinateness** for vague predicates.

Why sST Fails: Soritical Revenge

In sST, adding Grd generates a **revenge paradox**:

- In SK semantics, **conjunction elimination** holds universally:

$$\Gamma \models \mathcal{A} \wedge \mathcal{B}, \Delta \;\Rightarrow\; \Gamma \models \mathcal{A}, \Delta$$

- This makes conjunctions involving Grd **cuttable**.
- Using the tolerance principle with groundedness, Cut can be applied to derive $\vdash \bot$ — the theory **collapses**.

Bruni & Rossi (2023)

$sSTV$ cannot express determinateness — on pain of revenge paradox.

More specific explanation

(Grounded) Tolerance

If $P(c_i)$ is a true and grounded sentence and $c_i \sim_P c_{i+1}$, then $P(c_{i+1})$ is also true and grounded

- This is natural, but it leads to a **soritical revenge paradox**
- Let c_q be the last object with $v(P(c_q)) = 1$ and c_r be the first object with $v(P(c_r)) = n$. Then $P(c_q)$ is true and grounded, and $c_q \sim_P c_r$. The following Cut step go through:
- $\vdash P(c_q) \wedge Grd(P(c_q)) \wedge c_q \sim_P c_r$
  $P(c_q) \wedge Grd(P(c_q)) \wedge c_q \sim_P c_r \vdash P(c_r) \wedge Grd(P(c_r))$
- But $P(c_r) \wedge Grd(P(c_r))$ has value 0 in Strong Kleene.
- This means that $Grd(P(c_r)) \vdash \bot$ holds, and with Cut $\vdash \bot$

How wST Solves This

In WK semantics, **ungroundedness is infectious**:

$$v(P(c_k)) = n \;\Rightarrow\; v(Grd(\ulcorner P(c_k) \urcorner)) = 0$$

$$\Rightarrow\; v\big(P(c_k) \wedge Grd(\ulcorner P(c_k) \urcorner)\big) = n \quad \text{(not } 0 \text{ as in SK!)}$$

Proposition 3 (Ahmad 2025)

$P(c_k) \wedge Grd(\ulcorner P(c_k) \urcorner) \wedge c_k \sim_P c_{k+1}$ is **not cuttable** in $wSTVG$.

Proof sketch: Let $v(P(c_k)) = n$ and $v(Grd(\ulcorner P(c_k) \urcorner)) = 0$. By WK, the whole conjunction gets value n. So Cut fails: $\top \not\models_{wSTVG} \bot$.

$\Rightarrow$ The revenge argument is **blocked**.

What wST Preserves

Feature	**wST**
Transparent truth predicate	✓
Tolerance principle	✓
Classical recapture (grounded cases)	✓
Expressible groundedness / determinateness	✓
Revenge-free	✓
Conjunction elimination (grounded)	✓ (recoverable)

Key insight

Conjunction elimination is recoverable for *grounded* conjunctions. It only fails when an ungrounded conjunct is involved — exactly the right place for it to fail.