Vagueness and Base Logics (Fine & Holliday)

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Readings

- Fine, K. (2020). *Vagueness: A Global Approach*. Oxford University Press.
- Fine, K. (2017). “The Possibility of Vagueness.” *Synthese*, 194(10), 3699–3725.
- Holliday, W. H. (2025). “Vagueness and the Connectives.” In J. Yan, M. Liu, D. Westerståhl, and X. Yang (eds.), *The Connectives in Logic and Language*, 1–22. Springer.

Local vs Global

- **Local** indeterminacy concerns **whether** p_i.
- **Global** indeterminacy concerns **a range** $P = \langle p_1, \ldots, p_n \rangle$: whether the range admits a complete and correct answer.
- A localization principle takes the form: if P is globally indeterminate, then some p_i is locally indeterminate.
- In recent work, Fine (2020) rejects this principle.
- The task is thus to permit the global claim $\neg \bigwedge_{i=1}^{m} (p_i \vee \neg p_i)$ without inferring a local failure $\bigvee_{i=1}^{m} \neg(p_i \vee \neg p_i)$.

Determinacy and Indeterminacy

- Local determinacy is thus $\mathsf{D}(p) := p \vee \neg p$, while global determinacy is $\mathsf{D}(P) := \bigwedge_{i=1}^{m} \mathsf{D}(p_i)$
- Weak global indeterminacy is

$$\mathsf{I}(P) := \neg \mathsf{D}(P) = \neg \bigwedge_{i=1}^{m} (p_i \vee \neg p_i)$$

- Strong global indeterminacy is

$$\begin{aligned} \mathsf{I}^*(P) &:= \mathsf{I}(\neg p_1, \ldots, \neg p_m) \\ &= \neg \bigwedge_{i=1}^{m} (\neg p_i \vee \neg\neg p_i) \end{aligned}$$

Vagueness: endpoints without a boundary

- Let $P = \langle p_1, \ldots, p_m \rangle$ be a Sorites series.
- A **response** to P is an assignment of verdicts to cases in the series: yes, no, or a higher-order verdict.
- A **sharp response** is a complete response that draws a contrast somewhere in the series, possibly at a higher order.
- The **minimal response** records only the clear endpoint verdicts: the first case p_1 is positive, the last p_m is negative.
- **Compatibility:** Vagueness is compatible with the minimal endpoint response.
- So vagueness does not require denying the clear cases.
- **Incompatibility:** Vagueness is incompatible with every sharp response.
- So vagueness rules out any complete classification that draws a boundary.

The localist impossibility result

- No localist specification can both:
  1. allow the clear endpoints of a Sorites series and
  2. rule out every complete boundary-drawing response.
- If vagueness is specified locally, then each case can be classified by its local status: true, false, borderline, determinately true, determinately borderline, and so on.
- But these local statuses themselves form a complete classification of the series. So a boundary reappears at some order.
- Thus localism cannot have endpoints without a boundary.

A model is a space of compatible uses

A propositional model for Fine's compatibilist logic Cmp is

$$\mathcal{M} = \langle U, \|, \varphi \rangle,$$

where $U \neq \varnothing$, $\|$ is reflexive and symmetric, and $\varphi : \mathsf{At} \to \mathcal{P}(U)$.

- Atoms, conjunction, and disjunction are evaluated locally:

$$\begin{aligned} u \models p &\iff u \in \varphi(p) \\ u \models B \wedge C &\iff u \models B \text{ and } u \models C \\ u \models B \vee C &\iff u \models B \text{ or } u \models C \end{aligned}$$

- The conditional is relational:

$$\begin{aligned} u \models B \supset C \iff & (u \models B \text{ and } u \models C) \\ & \text{or } \forall v\big((u \parallel v \text{ and } v \models B) \Longrightarrow v \models C\big) \end{aligned}$$

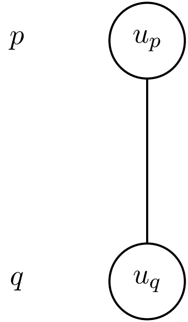
Negation looks across compatible uses

With $\neg B := B \supset \bot$, the derived clause is

$$u \models \neg B \quad \iff \quad \text{no use compatible with } u \text{ verifies } B.$$

- Hence $u \not\models B$ does not imply $u \models \neg B$: a compatible use may still verify B.

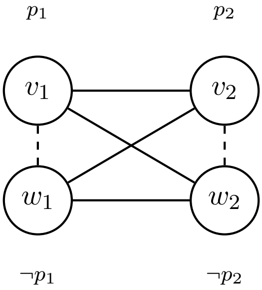
	$\mathsf{D}(p)$	$\mathsf{D}(q)$
u_p	$+$	$-$
u_q	$-$	$+$



- So no point verifies $\mathsf{D}(p) \wedge \mathsf{D}(q)$.
- Thus $\mathsf{I}(p, q)$ is verified at both points.
- Although no single $\mathsf{I}(p)$ is verified.

The Possibility Theorem: three-case illustration

- Let $P = \langle p_1, p_2, p_3 \rangle$, and use $U = \{v_1, w_1, v_2, w_2\}$.
- Set $\varphi(p_1) = \{v_1\}$, $\varphi(p_2) = \{v_2\}$, and $\varphi(p_3) = \varnothing$.
- Every point compatible with v_1 falsifies at least one conjunct of $\bigwedge_{i=1}^{3}(\neg p_i \vee \neg\neg p_i)$.
- Therefore $v_1 \models \mathsf{I}^*(P)$
- Moreover, $v_1 \models p_1$, and since no point verifies p_3, $v_1 \models \neg p_3$.



No cutoff does not entail Tolerance

- For adjacent members of a Sorites series, compare
  - $\mathrm{Tol}_i := p_i \supset p_{i+1}$
  - $\mathrm{Cut}_i := \neg(p_i \wedge \neg p_{i+1})$
- In classical logic the formulas are equivalent. Here they are not:
  - $p_i, \mathrm{Tol}_i \models_{\mathsf{Cmp}} p_{i+1}$
  - $p_i, \mathrm{Cut}_i \not\models_{\mathsf{Cmp}} p_{i+1}$
- Fine can therefore reject Tolerance as a premise while retaining the weaker claim that there is no acceptable cutoff.

Holliday's framework

- Fine's compatibility logic $\mathtt{Cmp}$ makes the Sorites consistent by weakening classical logic.
- In recent work, Holliday (2025) distinguishes three possible base logics:
  - **Fine's $\mathtt{Cmp}$:** retains distributivity, but rejects excluded middle and double-negation elimination.
  - **Orthologic:** retains excluded middle, but rejects distributivity.
  - **Fundamental logic:** rejects both distributivity and excluded middle.
- Thus Holliday asks whether Fine's rejection of excluded middle is essential, or whether rejecting distributivity already suffices.

Same compatibility picture, different disjunction

- Fine uses models $\mathcal{M} = \langle U, \|, \varphi \rangle$, where $\|$ is reflexive and symmetric.
- Holliday uses relational models $\mathcal{H} = \langle U, \|, \varphi \rangle$, but requires each $\varphi(p)$ to satisfy a fixpoint condition.
- Negation has the same relational clause: $u \models \neg B$ iff no $v \parallel u$ verifies B.
- Conjunction is also evaluated in the same way.
- The main difference concerns disjunction:
  - in Fine's Cmp, $u \models B \vee C$ iff $u \models B$ or $u \models C$.
  - in Holliday's framework, $B \vee C$ is obtained by closing the union of the propositions expressed by B and C.
- Consequently, the two frameworks agree on the $\{\wedge, \neg\}$-fragment, but diverge once $\vee$ is involved.

Holliday and Fine

- The model can verify the endpoints p_1 and $\neg p_m$, while verifying $\neg(p_i \wedge \neg p_{i+1})$ for every adjacent pair.
- But the explanations differ:
  - With $\mathsf{D}(p_i) = p_i \vee \neg p_i$, Fine's compatibility logic does not validate excluded middle in general. For a vague range, Fine's claim is the global denial $\neg \bigwedge_i \mathsf{D}(p_i)$, not the denial of any particular $\mathsf{D}(p_i)$.
  - Orthologic validates each $\mathsf{D}(p_i) = p_i \vee \neg p_i$, but not the distributive inference $\bigwedge_i \mathsf{D}(p_i) \models \bigvee_{\varepsilon \in \{0,1\}^m} \bigwedge_i p_i^{\varepsilon_i}$.
- Fundamental logic goes further: it also allows $p_i \vee \neg p_i$ to fail.
- It can therefore express Fine's global claims: $\mathsf{I}(P) = \neg \bigwedge_i \mathsf{D}(p_i)$ and $\mathsf{I}^*(P) = \neg \bigwedge_i (\neg p_i \vee \neg\neg p_i)$.

Every Fine model has an orthologic core

- Fix a reflexive, symmetric Fine frame $F = \langle U, \| \rangle$.
- For $A \subseteq U$, let $A^{\perp} := \{u \in U : \text{no } v \parallel u \text{ lies in } A\}$ and $c(A) := A^{\perp\perp}$.
- Double negation is a closure:
  - $A \subseteq c(A)$
  - if $A \subseteq B$, then $c(A) \subseteq c(B)$
  - $c(c(A)) = c(A)$
- Its fixed points are the regular sets $\mathrm{Reg}(F) := \{A \subseteq U : c(A) = A\}$.
- For $A, B \in \mathrm{Reg}(F)$, put $A \sqcap B := A \cap B$, $\neg_R A := A^{\perp}$, and $A \sqcup B := c(A \cup B)$.
- These operations make $\mathrm{Reg}(F)$ a complete ortholattice. Its join is the **closure of a union**, not Fine's pointwise $\vee$.

Double negation

- On atoms and negations, define $g(p) := \neg\neg p$ and $g(\neg\varphi) := \neg g(\varphi)$.
- On the binary connectives, define $g(\varphi \wedge \psi) := g(\varphi) \wedge g(\psi)$ and $g(\varphi \vee \psi) := \neg(\neg g(\varphi) \wedge \neg g(\psi))$.
- Given any Fine valuation V, regularize its atoms by $V^{\#}(p) := c(V(p))$.
- Induction on φ gives the exact model identity $[\![g(\varphi)]\!]_V^{\mathrm{Fine}} = [\![\varphi]\!]_{V^{\#}}^{\mathrm{OL}}$.
- At the disjunction step, for translated extensions A, B, $(A^{\perp} \cap B^{\perp})^{\perp} = (A \cup B)^{\perp\perp} = c(A \cup B) = A \sqcup B$.
- Note that in general c need not preserve intersections.

Orthologic is preserved frame by frame

- For each fixed reflexive, symmetric frame F, $F \models_{\mathrm{Fine}} g(\varphi) \Rightarrow g(\psi)$ if and only if $F \models_{\mathrm{OL}} \varphi \Rightarrow \psi$.
- (Fine $\rightarrow$ orthologic). Every Fine valuation V produces the closed valuation $V^{\#} = c \circ V$. The model identity transfers validity to the translated Fine formulas.
- (Orthologic $\rightarrow$ Fine). Every orthologic valuation W is already closed, so $W^{\#} = W$. Hence every orthological countermodel remains available after translation.